Online Appendix for "Dynamic Matching in School Choice: Efficient Seat Reassignment after Late Cancellations"

A List of Notation

A.1 Model

- $S = \{s_1, \ldots, s_N\}$: schools
- s_{N+1}: outside option
- q_i: capacity of school i
- $\Lambda = \Theta \times [0, 1]$: set (continuum) of students
- η: measure over Λ
- $\theta = (\succ^\theta, \hat{\succ}^\theta, p^\theta)$: student types
- Θ: space of student types θ
- $\zeta(\theta)$: measure of students with type θ
- L: student lottery numbers
- n_i: the number of priority groups at school s_i

A.2 Mechanisms

- P: permutation
- μ: first-round assignment
- $\hat{\mu}$: second-round assignment
- $\hat{\mu}_P$: second-round assignment from PLDA with permutation P
- $\mathbf{C}$: first-round cutoffs
- $\hat{\mathbf{C}}^P$: second-round cutoffs from PLDA with permutation P
- $\mathbf{C}_\pi$: first-round cutoffs restricted to priority class π

- $\hat{\mathbf{C}}^P_\pi$: second-round cutoffs from PLDA with permutation P restricted to priority class π

A.3 Proof of Theorem 2

- $\ell_P = \eta(\{\lambda \in \Lambda : \theta^\lambda = \theta, \mu(\lambda) = s_i, \hat{\mu}_P(\lambda) \neq s_i\})$: the measure of students with type θ leaving school s_i in the second round under PLDA with permutation P

- $e_P = \eta(\{\lambda \in \Lambda : \theta^\lambda = \theta, \mu(\lambda) \neq s_i, \hat{\mu}_P(\lambda) = s_i\})$: the measure of students with type θ entering school s_i in the second round under PLDA with permutation P

A.4 Proofs for Uniform Dropouts (Section 4)

- ρ: probability that a student drops out ac

- $\tilde{\mathbf{C}}$: constructed second-round cutoffs

- $f_i^P(x)$: proportion of students with s_i in their affordable set with permutation P and first- and second-round cutoffs (C_i, x)

- γ_i^P: the fraction of students whose affordable set in the second round of PLDA with permutation P is X_i

A.5 Notation in the Appendix

- $\hat{r}_i^\lambda = P(L(\lambda)) + n_i \mathbb{1}_{\{L(\lambda) \geq C_i\}} + p_i^\lambda \mathbb{1}_{\{L(\lambda) < C_i\}}$: the amended second-round score of student λ under PLDA

- $X_i = \{s_i, \ldots, s_{N+1}\}$: schools (weakly) after s_i in the cutoff ordering

- γ_i: the proportion of students whose first-round affordable set is X_i

A.6 Proof of Theorems 1 and 4

- $\beta_{i,j} = \eta(\{\lambda \in \Lambda : \operatorname{argmax}_{\succ^\lambda} X_j = s_i\})$: the measure of students who, when their set of affordable schools is X_j, will choose s_i

- $E^\lambda(\mathbf{C})$: the set of schools affordable for type λ in the first round under PLDA with permutation P

- $\hat{E}^{\lambda}(\hat{\mathbf{C}}^{P})$: the set of schools affordable for type λ in the second round under PLDA with permutation P
- $\gamma_i^P = \eta(\{\lambda \in \Lambda : \hat{E}_P^{\lambda}(\hat{\mathbf{C}}^P) = X_i\})$: the fraction of students whose affordable set in the second round of PLDA with permutation P is X_i
- q_π: restricted capacity vector for priority class π
- Λ_π: set of students with priority class π
- η_π: restriction of η to students with priority class π
- $\mathcal{E}_\pi = (S, q_\pi, \Lambda_\pi, \eta_\pi)$: restricted primitives for priority class π
- $s_{\sigma_\pi(i)}$: i-th school under second-round overdemand ordering for $\mathcal{E}_\pi$
- $\tilde{\mathbf{C}}^P$: second-round cutoffs defined for PLDA with the amended second-round scores from the RLDA cutoffs $\hat{\mathbf{C}}^R$
- $\tilde{\mathbf{C}}_\pi^P$: second-round cutoffs defined for PLDA on $\mathcal{E}_\pi$ with the amended second-round scores from the RLDA cutoffs $\hat{\mathbf{C}}^R$
- $\hat{n}$: smallest index of a school affordable to everyone

A.7 Proof of Theorem 3

- $s_{\sigma(i)}$: i-th school under second-round overdemand ordering in a non-atomic mechanism M satisfying axioms (1)–(5)
- $\tilde{X}_i = \{s_{\sigma(i)}, s_{\sigma(i+1)}, \dots, s_{\sigma(N+1)}\}$: schools (weakly) after $s_{\sigma(i)}$ in the second-round overdemand ordering
- $\gamma_{i,j}$: proportion of students under the constructed PLSM whose first-round affordable set was X_i and whose second-round affordable set was $\tilde{X}_j$
- $i(S') = \max\{j \ : \ s_j \in S'\}$: the maximum index of a school in S'
- $I_i^j = [C_i, C_j]$: the first-round scores that give students first-round affordable sets $\{X_{j+1}, X_{j+2}, \dots, X_i\}$
- $\rho^\theta(I, S')$: the proportion of students with type θ who, under the mechanism M, have a first-round score in the interval I and are assigned to a school in S' in the second round

B PLDA for a Discrete Set of Students

In this section we formally define and show how to implement PLDA mechanisms in a discrete setting with a finite number of students, and prove that they retain almost all the desired incentive and efficiency properties discussed in Section 2.2.

B.1 Discrete Model

A finite set $\Lambda = \{1, 2, \ldots, n\}$ of students are to be assigned to a set $S = \{s_1, \ldots, s_N\}$ of schools. Each student can attend at most one school. As in the continuum model, for every school $s_i \in S$, let $q_i \in \mathbb{N}_+$ be the *capacity* of school s_i, i.e., the number of students the school can accommodate. Let $s_{N+1} \notin S$ denote the outside option, and assume $q_{N+1} = \infty$. For each set of students $A \subseteq \Lambda$ we let $\eta(A) = |A|$ be the number of students in the set. As in the continuum model, each student $\lambda = (\theta^\lambda, L(\lambda)) \in \Lambda$ has a type $\theta^\lambda = (\succ^\lambda, \hat{\succ}^\lambda, p^\lambda)$ and a first-round lottery number $L(\lambda) \in [0, 1]$, which encode both student preferences and school priorities. The first-round lottery numbers $L(\lambda)$ are i.i.d. random variables drawn uniformly from $[0, 1]$ and do not depend on preferences. These random lottery numbers L generate a uniformly random permutation of the students based on the order of their lottery numbers.

An *assignment* $\mu : \Lambda \to S$ specifies the school that each student is assigned to. For an assignment μ, we let $\mu(\lambda)$ denote the school to which student λ is assigned, and in a slight abuse of notation, we let $\mu(s_i)$ denote the *set* of students assigned to school s_i. As in the continuum model, we say that a student $\lambda \in \Lambda$ is a *reassigned student* if she is assigned to a school in S in the second round that is different to her first-round assignment.

B.2 PLDA Mechanisms & Their Properties

We now formally define PLDA mechanisms in a setting with a finite number of students. In order to do so, we use the algorithmic description of DA and extend it to a two-round setting. This also provides a clear way to implement PLDA mechanisms in practice.

We first reproduce the celebrated and widely deployed DA algorithm, and then proceed to define PLDAs.

Definition 1. *The Deferred Acceptance algorithm with single tie-breaking is a function* $DA\big((\succ^\lambda, p^\lambda)_{\lambda\in\Lambda}, L\big)$ *mapping the student preferences in the first round, priorities and lottery numbers into* *an assignment μ constructed as follows. In each step, unassigned students apply to their most-preferred school that has not yet rejected them. A school with a capacity of q* tentatively accepts *its q highest-ranked applicants, ranked according to its priority ranking of the students with ties broken by giving preference to higher lottery numbers L (or tentatively accepts all applicants, if fewer than q have applied), and rejects any remaining applicants, and the algorithm moves on to the next step. The algorithm runs until there are no new student applications, at which point it terminates and assigns each student to her tentatively assigned school seat.*

Definition 2 (Permuted Lottery Deferred Acceptance (PLDA) mechanisms)**.** *Let P be a permutation of Λ. Let L be the realization of first-round lottery numbers, and let μ be the first-round assignment obtained by running DA with lottery L. The permuted lottery deferred acceptance mechanism associated with P (PLDA(P)) is the mechanism that then computes a second-round assignment $\hat{\mu}^P$ by running DA on the same set of students Λ but with student preferences $\hat{\succ}$, a modified lottery $P \circ L$, and modified priorities $\hat{p}$ that give each student top priority at the school she was assigned to in the first round. Specifically, each school s_i's priorities $\hat{\succ}_i$ are defined by lexicographically ordering the students first by whether they were assigned to s_i in the first round, and then according to p_i. PLDA(P) is the two-round mechanism obtained from using the reassignment mechanism $DA\big((\hat{\succ}^\lambda, \hat{p}^\lambda)_{\lambda\in\Lambda}, P \circ L\big)$.*

We now formally define desirable properties from Section 2.2 in our discrete model. We remark that the definitions of respecting guarantees, strategy-proofness and anonymity do not reference school capacities and so carry over immediately. Similarly, the definitions for respecting priorities, non-wastefulness and constrained Pareto efficiency do not require non-atomicity and so our definition of η ensures that they also carry over. For completeness, we rewrite these properties without reference to η.

Definition 3. *A two-round mechanism M **respects priorities (subject to guarantees)** if (i) for every school $s_i \in S$ and student $\lambda \in \Lambda$ who prefers s_i to her assigned school $s_i \hat{\succ}^\lambda \hat{\mu}(\lambda)$, we have $|\hat{\mu}(s_i)| = q_i$, and (ii) for all students λ' such that $\hat{\mu}(\lambda') = s_i \neq \mu(\lambda')$, we have $p_i^{\lambda'} \geq p_i^\lambda$.*

Definition 4. *A two-round mechanism is **non-wasteful** if no student is denied a seat at a school that has vacant seats; that is, for each student $\lambda \in \Lambda$ and school s_i, if $s_i \hat{\succ}^\lambda \hat{\mu}(\lambda)$, then $|\hat{\mu}(s_i)| = q_i$.*

Let $\hat{\mu}$ be a second-round assignment. A *Pareto-improving cycle* is an ordered set of students $(\lambda_1, \lambda_2, \dots, \lambda_m) \in \Lambda^m$ and schools $(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_m) \in S^m$ such that $\tilde{s}_{i+1} \hat{\succ}^i \tilde{s}_i$ (where $\hat{\succ}^i$ denotes the second-round preferences of student λ_i, and we define $\tilde{s}_{m+1} = \tilde{s}_1$), and $\hat{\mu}(\lambda_i) = \tilde{s}_i$ for all i.

Let $\hat{p}$ be the second-round priorities obtained by giving each student λ a top second-round priority $\hat{p}_i^\lambda = n_i$ at their first-round assignment $\mu(\lambda) = s_i$ (if $s_i \in S$) and unchanged priority $\hat{p}_j^\lambda = p_j^\lambda$ at all other schools $s_j \neq s_i$. We say that a Pareto-improving cycle (in a second-round assignment) *respects (second-round) priorities* if $\hat{p}_{\tilde{s}_{i+1}}^{\lambda_i} \geq \hat{p}_{\tilde{s}_{i+1}}^{\lambda_{i+1}}$ for all i (where we define $\lambda_{m+1} = \lambda_1$).

Definition 5. *A two-round mechanism is* ***constrained Pareto efficient*** *if the second-round assignment has no Pareto-improving cycles that respect second-round priorities.*

In a setting with a finite number of students, PLDA mechanisms exactly satisfy all these properties except for strategy-proofness.

Proposition 1. *PLDA mechanisms respect guarantees and priorities, and are non-wasteful, constrained Pareto efficient, and anonymous.*

Proof. The proofs of all these properties are almost identical to those in the continuum setting.

As an illustration, we prove that PLDA is constrained Pareto efficient in the discrete setting by using the fact that both rounds use single tie-breaking and the output is stable with respect to the second-round priorities $\hat{p}$.

Fix a Pareto-improving cycle C. Since λ_i is assigned a seat at a school s_i when she prefers (according to her second-round preferences) $s_{i+1} = \mu(\lambda_{i+1})$, by the stability of DA she must either be in a strictly worse (second-round) priority group than λ_{i+1} at school s_{i+1}, or in the same priority group but have a worse lottery number. If C respects (second-round) priorities, then it must hold that for all i that students λ_i and λ_{i+1} are in the same priority group at school s_{i+1} and λ_i has a worse lottery number than λ_{i+1}. But since this holds for all i, single tie-breaking implies that we obtain a cycle of lottery numbers, which provides the necessary contradiction. □

Proposition 2 states that in a setting with a finite number of students, PLDA mechanisms satisfy all our desired properties except for strategy-proofness. The following example illustrates that in a setting with a finite number of students, PLDA mechanisms may not satisfy two-round strategy-proofness. The intuition is that without non-atomicity, students are able to manipulate the first-round assignments of other students to change the guarantees, and hence change the second-round stability structure. In some cases in small markets, students are able to change the set of stable outcomes to benefit themselves.

Example 1 (PLDA with finite number of students is not strategy-proof.). *Consider a setting with $N = 2$ schools and $n = 4$ students. Each school has capacity 1 and a single priority class. For readability, we let $\emptyset$ denote the outside option, $\emptyset = s_{N+1} = s_3$. The students have the following preferences:*

1. $s_1 \succ_1 \emptyset \succ_1 s_2$ *and* $\emptyset \mathrel{\hat{\succ}_1} s_1 \mathrel{\hat{\succ}_1} s_2$,
2. $s_1 \succ_2 s_2 \succ_2 \emptyset$, *second-round preferences identical,*
3. $s_2 \succ_3 s_1 \succ_3 \emptyset$, *second-round preferences identical,*
4. $s_2 \succ_4 \emptyset \succ_4 s_1$, *second-round preferences identical.*

We show that the two-round mechanism where the second round is the reverse lottery deferred acceptance mechanism is not strategy-proof.

Consider the lottery that yields $L(1) > L(2) > L(3) > L(4)$. If the students report truthfully, the first-round assignment and second-round reassignment are

$$\begin{aligned} \mu(\Lambda) &= (\mu(1), \mu(2), \mu(3), \mu(4)) = (s_1, s_2, \emptyset, \emptyset), \text{ and} \\ \hat{\mu}(\Lambda) &= (\hat{\mu}(1), \hat{\mu}(2), \hat{\mu}(3), \hat{\mu}(4)) = (\emptyset, s_2, s_1, \emptyset) \end{aligned}$$

respectively. However, consider what happens if student 2 says that only school 1 is acceptable to her by reporting preferences $\succ^{\mathrm{r}}, \hat{\succ}^{\mathrm{r}}$ given by $s_1 \succ^{\mathrm{r}} \emptyset \succ^{\mathrm{r}} s_2$ and $s_1 \hat{\succ}^{\mathrm{r}} \emptyset \hat{\succ}^{\mathrm{r}} s_2$. Then

$$\mu(\Lambda) = (s_1, \emptyset, s_2, \emptyset), \quad \hat{\mu}(A) = (\emptyset, s_1, s_2, \emptyset),$$

which is a strictly beneficial change for student 2 in the second round (and, in fact, weakly beneficial for all students).

Note that this reassignment was not stable in the second round when students reported truthfully, since, in that case, school s_2 had second-round priorities $p_2^2 > p_2^4 > p_2^3 > p_2^1$ and so school s_2 and student 4 formed a blocking pair. In other words, for this particular realization of lottery numbers, student 2 is able to select a beneficial second-round assignment $\hat{\mu}$ that was previously unstable by changing student 3's first-round assignment so that student 4 cannot block $\hat{\mu}$.

In addition, the second-round outcome for student 2 under misreporting stochastically dominates her outcome from truthful reporting, when all other students report truthfully and the randomness is due the first-round lottery order. For if the lottery order is $L(1) > L(2) > L(3) > L(4)$ then *student 2 can change her second-round assignment from s_2 to s_1 by reporting s_2 as unacceptable, and this is the only lottery order for which student 2 receives a second-round assignment of s_2 under truthful reporting.*[1] *Moreover, for any lottery order where student 2 received s_1 in the first or second round under truthful reporting, she also received s_1 in the same round by misreporting.*[2] *Hence the second-round assignment student 2 receives by misreporting stochastically dominates the assignment she would have received under truthful reporting. This violates strategy-proofness.*

This example shows that, as noted in Section 6, PLDA mechanisms are not two-round strategy-proof in the finite setting. However, there are convergence results in the literature that suggest that PLDA mechanisms are almost two-round strategy-proof in large markets. We conjecture that the proportion of students who are able to successfully manipulate PLDA mechanisms decreases polynomially in the size of the market; a formal proof of such a result is beyond the scope of this paper.

Moreover, we believe that students will be unlikely to try to misreport under the PLDA mechanisms.[3] This is because, as Example 2 illustrates, successful manipulations require that students strategically change their first-round assignment and correctly anticipate that this changes the set of second-round stable assignments to their benefit. Such deviations are very difficult to plan and require sophisticated strategizing and detailed information about other students' preferences.

C Proofs

We begin with some general notation and definitions. Let μ be the initial assignment under DA-STB, and let P be a permutation. We say that a school s_i *reaches capacity* under a mechanism with output assignment μ if $\eta(\mu(s_i)) = q_i$.

We re-index the schools in $S \cup \{s_{N+1}\}$ so that $C_i \geq C_{i+1}$. Moreover, we assume that this indexing is done such that if the order condition is satisfied, then $\hat{C}_i^P \geq \hat{C}_{i+1}^P$ (where the cutoffs $\hat{\mathbf{C}}^P$ are as defined by PLDA(P)) holds simultaneously for all permutations P.

Recall that in DA each student is given a score $r_i^\lambda = p_i^\lambda + L(\lambda)$, and in PLDA$(P)$ this leads to

[1] This is because if $L(\lambda) > L(2)$ for $\lambda = 3, 4$ then student λ is assigned to school s_2 and stays there in both rounds, if $L(2) > L(1), L(3), L(4)$ then student 2 is assigned to school s_1 and stays there in both rounds, and finally if $L(1) > L(2) > L(4) > L(3)$ then student 2 is assigned to s_1 in the second round.

[2] This is because any stable matching in which student 2 is assigned s_1 remains stable after student 2 truncates. Indeed, student 2 is not part of any unstable pair, as she got her first choice, and any unstable pair not involving student 2 remains unstable under the true preferences, as only student 2 changes her preferences.

[3] as compared to the currently used DA mechanism.

a second-round score $\hat{r}_i^\lambda = \hat{p}_i^\lambda + P(L(\lambda)) = P(L(\lambda)) + n_i \mathbb{1}_{\{\mu(\lambda)=s_i\}} + p_i^\lambda \mathbb{1}_{\{\mu(\lambda)\neq s_i\}}$. Throughout the Appendix, for convenience, we slightly change the second-round score of a student λ under PLDA with permutation P to be $\hat{r}_i^\lambda = P(L(\lambda)) + n_i \mathbb{1}_{\{L(\lambda)\geq C_i\}} + p_i^\lambda \mathbb{1}_{\{L(\lambda)<C_i\}}$, meaning that we give each student a guarantee at any school for which she met the cutoff in the first round. By consistency of preferences, it is easily seen that this has no effect on the resulting assignment or cutoffs.

We say that a student can *afford* a school in a round if her score in that round is at least as large as the school's cutoff in that round. We say that the set of schools a student can afford in the second round (with her amended second-round score) is her *affordable set.*

Throughout the Appendix, we let $X_i = \{s_i, \ldots, s_{N+1}\}$ be the set of schools at least as affordable as school s_i, and we let γ_i be the proportion of students whose first-round affordable set is X_i.

C.1 Proof of Proposition 1

Fix a permutation P and some PLDA with permutation P. We show that this particular PLDA satisfies all the desired properties. Let η be a distribution of students, and let $\hat{\mathbf{C}}^P$ be the second-round cutoffs corresponding to the assignment given by the PLDA for this distribution of student types. For now we relax the assumptions that all students have consistent preferences and that consistent types have full support.

PLDA respects guarantees because fewer students are guaranteed at each school than the capacity of the school. PLDA is non-wasteful because the second round terminates with a stable matching where all schools find all students acceptable, which is non-wasteful.

PLDA is constrained Pareto efficient, since we use single tie-breaking and the output is the student-optimal stable matching with respect to the updated second-round priorities $\hat{p}$.

This is easily seen via the cutoff characterization. Let the second-round cutoffs be $\hat{P}$, where overloading notation we let $\hat{P}_i$ denote the cutoff for school $\tilde{s}_i$. Fix a Pareto-improving cycle $(\Theta^m, \Lambda^m, S^m)$. Without loss of generality we may assume that $\hat{p}_{\tilde{s}_i}^\lambda + L(\lambda) \geq \hat{P}_i$ for all $\lambda \in \Lambda_i$, since the set of students for whom this is not true has measure 0. Moreover, since all students $\lambda \in \Lambda_i$ prefer (according to their second-round preferences) school $\tilde{s}_{i+1}$ to their assigned school $\hat{\mu}(\lambda) = \tilde{s}_i$, without loss of generality we may also assume that $\hat{p}_{\tilde{s}_{i+1}}^\lambda + L(\lambda) < \hat{P}_{i+1}$ for all $\lambda \in \Lambda_i$, since the set of students for whom this is not true has measure 0. This means that for all $\lambda_i \in \Lambda_i$ and $\lambda_{i+1} \in \Lambda_{i+1}$ it holds that $\hat{p}_{\tilde{s}_{i+1}}^{\lambda_i} + L(\lambda_i) < \hat{P}_{i+1} \leq \hat{p}_{\tilde{s}_{i+1}}^{\lambda_{i+1}} + L(\lambda_{i+1})$, and so $\hat{p}_{\tilde{s}_{i+1}}^{\lambda_i} \leq \hat{p}_{\tilde{s}_{i+1}}^{\lambda_{i+1}}$.

Suppose for the sake of contradiction that the cycle $(\Theta^m, \Lambda^m, S^m)$ respects second-round priorities. Then for each $\lambda_i \in \Lambda_i$ and $\lambda_{i+1} \in \Lambda_{i+1}$ it holds that $\hat{p}^{\lambda_i}_{\tilde{s}_{i+1}} \geq \hat{p}^{\lambda_{i+1}}_{\tilde{s}_{i+1}}$, and so $L(\lambda_i) > L(\lambda_{i+1})$. But since this holds for all i we obtain a cycle of lottery numbers $L(\lambda_1) > L(\lambda_2) > \cdots > L(\lambda_m) > L(\lambda_1)$ for all $\lambda_i \in \Lambda_i$, which provides the necessary contradiction.

Suppose now that student preferences are consistent. We show that the PLDA mechanism is strongly two-round strategy-proof. Since students are non-atomic, no student can change the cutoffs $\hat{\mathbf{C}}^P$ by changing her first- or second-round reports. Hence it is a dominant strategy for all students to report truthfully in the second round. Moreover, for any student of type λ, the only difference between having a first-round guarantee at a school s_i and having no first-round guarantee is that in the former case, $\hat{r}^\lambda_i$ increases by $n_i - p^\lambda_i$. This means that having a guarantee at a school s_i changes the student's second-round assignment in the following way. She receives a seat in school s_i whereas without the guarantee she would have received a seat in some school s_j that she reported preferring less to s_i, and her second-round assignment is unchanged otherwise. Therefore, students want their first-round guarantee to be the best under their second-round preferences, and so it is a dominant strategy for students with consistent preferences to report truthfully in the first round.

C.2 Proofs for Uniform Dropouts (Section 4)

In this section, we furnish proofs for the special case of uniform dropouts. The proofs techniques in this section mirror those used in the general setting and can give the interested reader a taste of the general proof techniques in a more transparent setting.

We show first that the global order condition (Definition 16) holds in the setting with uniform dropouts. The high level steps and algebraic tools used in this proof are similar to those used to show that the order condition is equivalent to the global order condition in our general framework (Theorem 4), although the analysis in each step is greatly simplified. Here we prove Theorem 5 and briefly outline the additional steps necessary to prove Theorem 4.

Proof of Theorem 5. The main steps in the proof are as follows: (1) *Assuming that every student's affordable set is X_i for some i,* for every school s_j, guess the proportion of students who should receive an affordable set that contains s_j. (2) Calculate the corresponding second-round cutoffs $\tilde{C}_j$ for school s_j. (3) Show that these cutoffs are in the same order as the first-round cutoffs. (4) Use the fact that the cutoffs are in the same order to verify that the cutoffs are market-clearing, and deduce that the constructed cutoffs are precisely the PLDA(P) cutoffs.

Throughout this proof, we amend the second-round score of a student λ under PLDA(P) to be

$\hat{r}_i^\lambda = P(L(\lambda)) + \mathbb{1}_{\{L(\lambda) \geq C_i\}}$, meaning that we give each student a guarantee at any school for which she met the cutoff in the first round. By consistency of preferences, it is easily seen that this has no effect on the resulting assignment or cutoffs. Let the first-round cutoffs be $C_1, C_2, \ldots, C_N$, where without loss of generality we index the schools such that $C_1 \geq C_2 \geq \cdots \geq C_N$.

(1) In the setting with uniform dropouts, since the second-round problem is a rescaled version of the first-round problem (with a $(1-\rho)$ fraction of the original students remaining), we guess that we want the proportion of students with an affordable set containing s_j to be $\frac{1}{1-\rho}$ times the original proportion. (In the general setting, we no longer have a rescaled problem and so we instead guess that the proportion of students with each affordable set is the same as that under RLDA.)

(2) We translate this into cutoffs. Let $f_i^P(x) = |\{l : l \geq C_i \text{ or } P(l) \geq x\}|$ be the proportion of students who receive school s_i in their (second-round) affordable set with the amended second-round scores under permutation P if the first- and second-round cutoffs are C_i and x respectively. Notice that $f_i(x)$ is non-increasing for all i, $f_i(0) = 1$, $f_i(1) = 1 - C_i$, and if $i < j$ then $f_i(x) \leq f_j(x)$ for all $x \in [0,1]$. Let the cutoff $\tilde{C}_i^P \in [0,1]$ be the minimal cutoff satisfying the equation $f_i(\tilde{C}_i^P) = \frac{1}{1-\rho}(1 - C_i)$, and let $\tilde{C}_i^P = 0$ if $C_i < \rho$. (In the general setting the cutoffs are defined using the same functions $f_i^P(\cdot)$ with the proportions being equal to those that arise under RLDA, as mentioned in step (1) above.)

(3) We now show that the cutoffs $\tilde{\mathbf{C}}$ are in the right order. Suppose that $i < j$. If $\tilde{C}_i^P = 0$ then $C_j \leq C_i \leq \rho$ and so $\tilde{C}_j^P = 0 \leq \tilde{C}_i^P$ as required. Hence we may assume that $\tilde{C}_i^P, \tilde{C}_j^P > 0$. In this case, since $f_j(\cdot)$ is non-increasing and $\tilde{C}_j^P$ is minimal, we can deduce that $\tilde{C}_j^P \leq \tilde{C}_i^P$ if $f_j\left(\tilde{C}_j^P\right) \geq f_j\left(\tilde{C}_i^P\right)$. It remains to establish the latter. Using the definition of f_j and f_i, we have

$$\begin{aligned} f_j\left(\tilde{C}_i^P\right) &= f_i\left(\tilde{C}_i^P\right) + \left|\left\{l : l \in [C_j, C_i),\ P(l) < \tilde{C}_i^P\right\}\right| \\ &\leq \frac{1}{1-\rho}(1 - C_i) + (C_i - C_j) \leq \frac{1}{1-\rho}(1 - C_j) = f_j\left(\tilde{C}_j^P\right), \end{aligned}$$

where both inequalities hold since $C_j \leq C_i$. It follows that $\tilde{C}_i^P \geq \tilde{C}_j^P$, as required. (In the general setting, since we cannot give closed form expressions for the proportions $f_i\left(\tilde{C}_i^P\right)$ in terms of the cutoffs C_i, this step requires using the intermediate value theorem and an inductive argument.)

(4) We now show that $\tilde{\mathbf{C}}^P$ is the set of market-clearing DA cutoffs for the second round of PLDA(P). Note that $\gamma_i = C_{i-1} - C_i$ is the proportion of students whose first-round affordable set is X_i (where $C_0 = 1$). Since dropouts are uniform at random, this is the proportion of such students out of the total number of remaining students both before and after dropouts.

Consider first the case $\tilde{C}_i^P > 0$. Now $f_i\left(\tilde{C}_i^P\right)$ is the proportion of students whose second-round affordable set contains s_i, and since $C_1 \geq C_2 \geq \cdots \geq C_N$ and $\tilde{C}_1^P \geq \tilde{C}_2^P \geq \cdots \geq \tilde{C}_N^P$, it follows that the affordable sets are nested. Hence the proportion of students (of those remaining after students drop out) whose second-round affordable set is X_i is given by (where $f_0(\cdot) \equiv 1$)

$$\gamma_i^P = f_i\left(\tilde{C}_i^P\right) - f_{i-1}\left(\tilde{C}_{i-1}^P\right) = \frac{C_{i-1} - C_i}{1-\rho} = \frac{\gamma_i}{1-\rho}.$$

For each $\theta = (\succ, \succ)$ and set of schools S, let $D^\theta(S)$ be the maximal school in S under $\succ$, and let $\theta' = (\succ, \hat{\succ})$ be the student type consistent with θ that finds all schools unacceptable in the second round. Then a set of students of measure

$$\sum_{j \leq i} \sum_{\theta \in \Theta : D^\theta(X_j) = i} \gamma_j^P \zeta(\theta) = \sum_{j \leq i} \gamma_j \sum_{\theta \in \Theta : D^\theta(X_j) = i} \frac{\zeta(\theta)}{1-\rho} = \sum_{j \leq i} \gamma_j \sum_{\theta \in \Theta : D^\theta(X_j) = i} \zeta(\theta) + \zeta(\theta')$$

choose to go to school s_i in the second round under the second-round cutoffs $\tilde{\mathbf{C}}^P$. We observe that the expression on the right gives the measure of the set of students who choose to go to school s_i in the first round under first-round cutoffs $\mathbf{C}$.

In the case where $\tilde{C}_i^P = 0$ the above expressions give upper bounds on the measure of the set of students who choose to go to school s_i in the second round under the second-round cutoffs $\tilde{\mathbf{C}}^P$. Since $\mathbf{C}$ are market-clearing cutoffs, and $\tilde{C}_i^P > 0 \Rightarrow C_i^P > 0$, it follows that $\tilde{\mathbf{C}}^P$ are market-clearing cutoffs too. We have shown that in PLDA(P), the second-round cutoffs are exactly the constructed cutoffs $\tilde{\mathbf{C}}^P$ and they satisfy $\tilde{C}_1^P \geq \cdots \geq \tilde{C}_N^P$, and so the global order condition holds. □

The general proof of Theorem 4 uses the cutoffs for RLDA in steps (1) and (2) above to guess the proportion of students who receive an affordable set that contains s_j, and requires that each student priority type be carefully accounted for. However, the general structure of the proof is similar, and the tools used are straightforward generalizations of those used in the proof above.

We now show that in the uniform dropouts case all PLDA mechanisms give type-equivalent assignments. We include the formal theorem statement and proof for this special case to illustrate the steps of the proof for the general case in a simpler setting.

The proof formalizes the intuition from Section 4 that the global order condition implies type equivalence, as it implies that the proportion of students of a given type who have a given affordable set does not depend on the permutation P, and the same insight is used in the proof of Theorem 1.

Lemma 1. *In any market with uniform dropouts (Definition 17), all PLDA mechanisms produce type-equivalent assignments.*

Proof. The proposition follows immediately from the fact that the proportion γ_i^P of students whose second-round affordable set is X_i does not depend on P. Specifically, consider first the case when all schools reach capacity in the second round of PLDA. We showed in the proof of Theorem 5 that for all i and all student types θ, the proportion of students of type θ with affordable set X_i in the second round under PLDA(P) is given by $\gamma_i^P = \frac{\gamma_i}{1-\rho}$, where γ_i is the proportion of students of type θ with affordable set X_i in the first round. It follows that all PLDAs are "type-equivalent" to each other because they are type-equivalent to the first-round assignment in the following sense. For each preference order $\succ$, let $\tilde{\succ}$ be the preferences obtained from $\succ$ by making the outside option the most desirable, i.e., $s_{N+1} \tilde{\succ} \cdots$. Then

$$\begin{aligned} \eta(\{\lambda \in \Lambda : \theta^\lambda = (\succ, \succ), \hat{\mu}^P(\lambda) = s_i\}) &= \frac{1}{1-\rho}\eta(\{\lambda \in \Lambda : \theta^\lambda = (\succ, \succ), \mu(\lambda) = s_i\}) \\ &= \eta(\{\lambda \in \Lambda : \theta^\lambda \in \{(\succ, \succ), (\succ, \tilde{\succ})\}, \mu(\lambda) = s_i\}), \end{aligned}$$

where the second equality holds since students stay in the system uniformly-at-random with probability $1 - \rho$. Under uniform dropouts this holds for all student types that remain in the system, and so it follows that $\hat{\mu}^P$ is type-equivalent to $\hat{\mu}^{P'}$ for all permutations P, P'.

When some school does not reach capacity in the second round, we can show by induction on the number of such schools that all PLDAs are type-equivalent to RLDA. □

C.3 Proof of Theorem 1

We first prove Theorem 1 in the case where all schools have one priority group. We then invoke Theorem 4 so that we may assume that the global order condition holds. We then show that this implies all PLDA mechanisms assign the same number of seats at a given school s_i to students of a given priority class π. Hence, by restricting to the set of students with priority class π, we can reduce the general problem to the case where all schools have one priority group. This shows that all PLDA mechanisms produce type-equivalent assignments.

Lemma 2. *Assume that each school has a single priority group,* $\mathbf{p} = \mathbf{1}$. *If the order condition holds, all PLDA mechanisms produce type-equivalent assignments.*

Proof. Let P be a permutation. Since the order condition holds, we can assume that the schools in

$S \cup \{s_{N+1}\}$ are indexed so that $C_i \geq C_{i+1}$ and $\hat{C}_i^P \geq \hat{C}_{i+1}^P$ for all permutations P (simultaneously).

We first present the relevant notation that will be used in this proof. We are interested in sets of schools of the form $X_i = \{s_i, \ldots, s_{N+1}\}$. Let

$$\beta_{i,j} = \eta(\{\lambda \in \Lambda : s_i \text{ is the most desirable school in } X_j \text{ with respect to } \hat{\succ}^\lambda\})$$

be the measure of the students who, when their set of affordable schools is X_j, will choose s_i (when following their second-round preferences). Note that $\beta_{i,j} = 0$ for all $j > i$.

Let $E^\lambda(\mathbf{C})$ and $\hat{E}_P^\lambda(\hat{\mathbf{C}}^P)$ be the sets of schools affordable for type λ in the first and second round, respectively, when running PLDA with lottery P. Note that for each student $\lambda \in \Lambda$, there exists some i such that $E^\lambda(\mathbf{C}) = X_i$, and since the order condition is satisfied, there exists some $j \leq i$ such that $\hat{E}_P^\lambda(\hat{\mathbf{C}}^P) = X_j$. The fact that $\hat{E}_P^\lambda(\hat{\mathbf{C}}^P) = X_j$ for some j is a result of the order condition: students' amended second-round scores guarantee that $E^\lambda(\mathbf{C}) \subseteq \hat{E}_P^\lambda(\hat{\mathbf{C}}^P)$ (every school affordable in the first round is guaranteed in the second) and hence that $j \leq i$. Let $\gamma_i^P = \eta(\{\lambda \in \Lambda : \hat{E}_P^\lambda(\hat{\mathbf{C}}^P) = X_i\})$ be the fraction of students whose affordable set in the second round of PLDA with permutation P is[4] X_i. We note that by definition of PLDA, $\eta(\{\lambda \in \Lambda : \theta^\lambda = \theta, \hat{E}_P^\lambda(\hat{\mathbf{C}}^P) = X_i\}) = \zeta(\{\theta\})\gamma_i^P$; that is, the students whose affordable sets are X_i "break proportionally" into types. For a school i, this means that the measure of students assigned to s_i is therefore $\sum_{j \leq i} \beta_{i,j}\gamma_j^P$.

Let P' be another permutation, and define $\gamma_i^{P'}$ similarly. We will prove by induction that there exist PLDA(P') cutoffs $\hat{\mathbf{C}}^{P'}$ such that $\gamma_i^{P'} = \gamma_i^P$ for all $s_i \in S \cup \{s_{N+1}\}$. Note that by the proportional breaking into types of γ_i^P and $\gamma_i^{P'}$, this will imply type-equivalence.

Assume that the PLDA(P') cutoffs $\hat{\mathbf{C}}^{P'}$ are chosen such that $\gamma_j^{P'} = \gamma_j^P$ for all $j < i$, and i is maximal such that this is true. Then we have that $\sum_{j \leq i-1} \beta_{i,j}\gamma_j^P = \sum_{j \leq i-1} \beta_{i,j}\gamma_j^{P'}$. Assume w.l.o.g. that $\gamma_i^P > \gamma_i^{P'}$. It follows that $q_i \geq \sum_{j \leq i} \beta_{i,j}\gamma_j^P \geq \sum_{j \leq i} \beta_{i,j}\gamma_j^{P'}$, where the first inequality follows since s_i cannot be filled beyond capacity. If the second inequality is strict, then under P', s_i is not full, and therefore $\hat{C}_i^{P'} = 0$. However, this means that all students can afford s_i under P', and therefore $\gamma_i^{P'} = 1 - \sum_{j<i} \gamma_j^{P'} = 1 - \sum_{j<i} \gamma_j^P \geq \gamma_i^P$, a contradiction. If the second inequality is an equality, then $\beta_{i,i} = 0$ and no students demand school i under the given affordable set structure. It follows that we can define the cutoff $\hat{C}_i^{P'}$ such that $\gamma_i^{P'} = \gamma_i^P$. This provides the required contradiction, completing the proof. $\square$

[4]Note that $\eta(\Lambda) = 1$, as η is a probability distribution over Λ.

Now consider when schools have possibly more than one priority group. We show that if the global order condition holds, then all PLDA mechanisms assign the same measure of students of a given priority type to a given school. It is not at all obvious that such a result should hold, since priority types and student preferences may be correlated, and the relative proportions of students of each priority type assigned to each school can vary widely. Nonetheless, global order condition imposes enough structure so that any given priority type is treated symmetrically across different PLDA mechanisms.

Theorem 1. *If the global order condition holds, then for all priority classes π and schools s_i all PLDA mechanisms assign the same measure of students of priority class π to school s_i.*

Proof. Fix a permutation P. We then show that PLDA(P) assigns the same measure of students of each priority type to each school s_i as RLDA. The idea will be to define cutoffs on priority-type-specific economies, and show that these cutoffs are the same as the PLDA cutoffs. However, since cutoffs are not necessarily unique in the two-round setting, care needs to be taken to make sure that the individual choices for priority-type-specific cutoffs are consistent across priority types.

The proof runs as follows. We first define an economy $\mathcal{E}_\pi$ for each priority class π that gives only as many seats as are assigned to students of priority class π under RLDA. We then invoke the global order condition and Theorems 4 and 1 to show that all PLDA mechanisms are type-equivalent on each $\mathcal{E}_\pi$. We also use the global order condition to argue that it is sufficient to consider affordable sets, and also to select "minimal" cutoffs. Then we construct cutoffs $\mathcal{C}^P_{\pi,i}$ using the economies $\mathcal{E}_\pi$ and show that they are (almost) independent of priority type. Finally, we show that this means that $\mathcal{C}^P_{\pi,i}$ also define PLDA cutoffs for the large economy $\mathcal{E}$ and conclude that PLDA(P) assigns the same measure of students of each priority type to each school s_i as RLDA.

(1) Defining little economies $\mathcal{E}_\pi$ for each priority type.

Fix a priority class π. Let q_π be a restricted capacity vector, where $q_{\pi,i}$ is the measure of students of priority class π assigned to school s_i under RLDA. Let Λ_π be the set of students λ such that $p^\lambda = \pi$, and let η_π be the restriction of the distribution η to Λ_π. Let $\mathcal{E}_\pi$ denote the primitives $(S, q_\pi, \Lambda_\pi, \eta_\pi)$. Recall that $\hat{\mathbf{C}^{\mathbf{R}}}$ are the second-round cutoffs for RLDA on $\mathcal{E}$. It follows from the definition of $\mathcal{E}_\pi$ that $\hat{\mathbf{C}}^R_\pi$ are also the second-round cutoffs for RLDA on $\mathcal{E}_\pi$.

Let $\tilde{\mathbf{C}}^P_\pi$ be the second-round cutoffs of PLDA(P) on $\mathcal{E}_\pi$. We show that the cutoffs $\tilde{\mathbf{C}}^P_\pi$ defined for the little economy are the same as the consistent second-round cutoffs $\hat{\mathbf{C}}^P_\pi$ for PLDA with permutation P for the large economy $\mathcal{E}$, that is, $\tilde{\mathbf{C}}^P_\pi = \hat{\mathbf{C}}^P_\pi$.

(2) Implications of the global order condition.

We have assumed that the global order condition holds.

This has a number of implications for PLDA mechanisms run on the little economies $\mathcal{E}_\pi$. For all p, the local order condition holds for RLDA on $\mathcal{E}_\pi$. Hence, by Theorem 4, the little economies $\mathcal{E}_\pi$ each satisfy the order condition. Moreover, by Theorem 1, all PLDA mechanisms produce type-equivalent assignments when run on $\mathcal{E}_\pi$. Finally, if we can show that for every permutation P, PLDA(P) assigns the same measure of students of each priority type π to each school s_i (namely $(q_\pi)_i$) as RLDA, then $\mathcal{E}$ satisfies the global order condition if and only if for all p the little economy $\mathcal{E}_\pi$ satisfies the global order condition.

The global order condition also allows us to determine aggregate student demand from the proportions of students who have each school in their affordable set. In general, if affordable sets break proportionally across types, and if for each subset of schools $S' \subseteq S$ we know the proportion of students whose affordable set is S', then we can determine aggregate student demand. The global order condition implies that for any pair of permutations P, P', the affordable sets from both rounds are nested in the same order under both permutations. In other words, for each priority class π there exists a permutation σ_π such that the affordable set of any student in any round of any PLDA mechanism is of the form $\{s_{\sigma_\pi(i)}, s_{\sigma_\pi(i+1)}, \ldots, s_{\sigma_\pi(N)}, s_{N+1}\}$. Hence when the global order condition holds, to determine the proportion of students whose affordable set is S', it is sufficient to know the proportion of students who have each school in their affordable set.

Another more subtle implication of the global order condition is the following. In the second round of PLDA, for each permutation P and school s_i there will generically be an interval that $\hat{C}_i^P$ can lie in and still be market-clearing. The intuition is that there will be large empty intervals corresponding to students who had school s_i in their first-round affordable set, and whose second-round lottery changed accordingly. When the global order condition holds, we can without loss of generality assume that as many as possible of the cutoffs for a given priority type are 0 or 1, and the global order condition will still hold.

Formally, for cutoffs C we can equivalently define priority-type-specific cutoffs $C_{\pi,i} = (\lfloor C_i - \pi_i \rfloor)^+$. Note the cutoffs C_π are consistent across priority types, namely: (1) Cutoffs match for two priority types with the same priority group at a school, $\pi_i = \pi'_i \Rightarrow C_{\pi,i} = C_{\pi',i}$ and $\hat{C}_{\pi,i} = \hat{C}_{\pi',i}$; and (2) There is at most one marginal priority group at each school, $C_{\pi,i}, C_{\pi',i} \in (0,1) \Rightarrow \pi_i = \pi'_i$. Moreover, if cutoffs C_π are consistent across priority types, then there exist cutoffs C from which they arise.

Suppose that we set as many as possible of the priority-type-specific cutoffs $\hat{C}^P_\pi$ to be extremal; i.e., we let $\hat{C}^P_{\pi,i}$ be 1 if no students have s_i in their affordable set, and let $\hat{C}^P_{\pi,i}$ be minimal otherwise. We show that under this new definition, $C_\pi, \hat{C}^P_\pi$ satisfies the local order condition consistently with all other PLDAs.

Specifically, let

$$f^P_{\pi,i}(x) = |\{l : l \geq C_{\pi,i} \text{ or } P(l) \geq x\}|$$

be the proportion of students of priority class π who have school s_i in their affordable set if the first- and second-round cutoffs are $C_{\pi,i}$ and x respectively. Notice that f is decreasing in x. Define cutoffs $\tilde{C}^P_\pi$ as follows. If $f^P_{\pi,i}\left(\hat{C}^P_{\pi,i}\right) = 0$ we set $\tilde{C}^P_{\pi,i} = 1$, and otherwise we let $\tilde{C}^P_{\pi,i}$ be the minimal cutoff satisfying $f^P_{\pi,i}\left(\tilde{C}^P_{\pi,i}\right) = f^P_{\pi,i}\left(\hat{C}^P_{\pi,i}\right)$.

Since $\mathcal{E}$ satisfies the global order condition, for all π there exists an ordering σ_π such that $C_{\sigma_\pi(1)} \geq C_{\sigma_\pi(2)} \geq \cdots \geq C_{\sigma_\pi(N)}$ and $\hat{C}^{P'}_{\sigma_\pi(1)} \geq \hat{C}^{P'}_{\sigma_\pi(2)} \geq \cdots \geq \hat{C}^{P'}_{\sigma_\pi(N)}$ for all permutations P'. We show that the global order condition implies that the newly defined cutoffs $\hat{C}^P$ satisfy $\tilde{C}^P_{\pi,\sigma_\pi(1)} \geq \tilde{C}^P_{\pi,\sigma_\pi(2)} \geq \cdots \geq \tilde{C}^P_{\pi,\sigma_\pi(n)}$. This is because the global order condition implies that f^P_π is increasing in i; i.e., for each π, $i < j$, and x it holds that $f^P_{\pi,\sigma_\pi(i)}(x) \leq f^P_{\pi,\sigma_\pi(j)}(x)$. Hence for all $j > i$,

$$\begin{aligned}
f^P_{\pi,\sigma_\pi(j)}\left(\tilde{C}^P_{\pi,\sigma_\pi(j)}\right) &= f^P_{\pi,\sigma_\pi(j)}\left(\hat{C}^P_{\pi,\sigma_\pi(j)}\right) \\
&\geq f^P_{\pi,\sigma_\pi(j)}\left(\hat{C}^P_{\pi,\sigma_\pi(i)}\right) \text{ (since } f \text{ is decreasing)} \\
&\geq f^P_{\pi,\sigma_\pi(i)}\left(\hat{C}^P_{\pi,\sigma_\pi(i)}\right) \text{ (since } f \text{ is increasing in } i) \\
&= f^P_{\pi,\sigma_\pi(i)}\left(\tilde{C}^P_{\pi,\sigma_\pi(i)}\right)
\end{aligned}$$

and so since we set $\tilde{C}^P_{\pi,\sigma_\pi(j)}$ to be minimal and $f^P_{\pi,\sigma_\pi(j)}(\cdot)$ is decreasing it follows that $\tilde{C}^P_{\pi,\sigma_\pi(j)} \leq \tilde{C}^P_{\pi,\sigma_\pi(i)}$.

(3) Cutoffs $\tilde{C}^P_{\pi,i}$ are (almost) independent of priority type.

We now show that $\tilde{C}^P_{\pi,i}$ depends on π only via π_i, and for all $j \neq i$ does not depend on π_j. Since $\mathcal{E}_\pi$ satisfies the order condition, all PLDA mechanisms on $\mathcal{E}_\pi$ are type-equivalent, and the proportion of students who have each school in their affordable set is the same across all PLDA mechanisms. Hence for all permutations P, priority classes π, and schools i it holds that $f^P_{\pi,i}\left(\tilde{C}^P_{\pi,i}\right) = f^P_{\pi,i}\left(\hat{C}^P_{\pi,i}\right) = f^R_{\pi,i}\left(\hat{C}^R_{\pi,i}\right)$. This means that $\tilde{C}^P_{\pi,i}$ satisfies the following equation in terms of $\hat{C}^R_{\pi,i}$, $C_{\pi,i}$

and P:

$$f^P_{\pi,i}\left(\tilde{C}^P_{\pi,i}\right) = f^R_{\pi,i}\left(\hat{C}^R_{\pi,i}\right) = 2 - \hat{C}^R_{\pi,i} - C_{\pi,i}. \tag{1}$$

(We note that an application of the intermediate value theorem shows that this equation always has a solution in $[0,1]$, since $f^P_{\pi,i}(0) = 1 - C_{\pi,i}$, $f_{\pi,i}(1) = 1$, $f_{\pi,i}$ is continuous and decreasing on $[0,1]$, and we are in the case where $1 - C_{\pi,i} \leq \hat{C}^R_{\pi,i} \leq 1$. Hence $\tilde{C}^P_{\pi,i}$ is defined by $f^P_{\pi,i}$ and $f^R_{\pi,i}$.) In other words, the value of $\tilde{C}^P_{\pi,i}$ is defined by $f^P_{\pi,i}(\cdot), f^R_{\pi,i}(\cdot)$, and $\hat{C}^R_{\pi,i}$, which in turn are defined by $C_{\pi,i}$ and the permutations P and R. Since $C_{\pi,i}$ depends on π only through π_i, it follows that $\tilde{C}^P_{\pi,i}$ depends on π only through π_i. In other words the $\tilde{C}^P_{\pi,i}$ define cutoffs $\tilde{C}^P_i$ that are independent of priority type.

(4) $\tilde{C}^P_i$ are the PLDA cutoffs.

Finally, we remark that $\tilde{C}^P_i$ are market-clearing cutoffs. This is because we have shown that for each priority class π, the number of students assigned to each school s_i is the same under RLDA and under the demand induced by the cutoffs $\tilde{C}^P_i$, and we know that the RLDA cutoffs are market-clearing for $\mathcal{E}$.

Hence $\tilde{C}^P_i$ give the assignments for PLDA on $\mathcal{E}$, and since $\tilde{C}^P_i$ was defined individually for each priority class π on $\mathcal{E}_\pi$, it follows that PLDA(P) assigns the same measure of students of each priority type to each school s_i as RLDA. □

We are now ready to prove Theorem 1

Proof of Theorem 1. Fix a priority class π. By Theorems 4 and 6, for every school s_i, all PLDA mechanisms assign the same measure $q_{\pi,i}$ of students of priority class π to school s_i.

Consider the subproblem with primitives $\mathcal{E}_\pi = (S, q_\pi, \Lambda_\pi, \eta_\pi)$. By Lemma 2, for all $\theta \in \Theta$ and s_i,

$$\eta_\pi(\{\lambda \in \Lambda_\pi : \theta^\lambda = \theta, \hat{\mu}_P(\lambda) = s_i\}) = \eta_\pi(\{\lambda \in \Lambda_\pi : \theta^\lambda = \theta, \hat{\mu}_{P'}(\lambda) = s_i\}).$$

Since η_π is the restriction of η to λ_π, it follows that all PLDA mechanisms are type-equivalent. □

C.4 Proof of Theorem 4

Note. The proof of Theorem 4 uses many of the same techniques as the proof for Theorem 5 the case of uniform dropouts. The interested reader looking to understand our proof techniques in a simpler setting can refer to Appendix C.2 for full proofs for this special case.

Proof of Theorem 4. Suppose that the order condition holds. In what follows, we will fix a permutation P and show that the PLDA mechanism with permutation P satisfies the local order condition and is type-equivalent to the reverse lottery RLDA mechanism. As this holds for every P, it follows that the global order condition holds.

(1) Every school has a single priority group.

We first consider the case where $n_i = 1$ for all i; that is, every school has a single priority group. Recall that the schools are indexed according to the first-round overdemand ordering, so that $C_1 \geq C_2 \geq \cdots \geq C_N \geq C_{N+1}$. Since the local order condition holds for RLDA, let us assume that they are also indexed according to the second-round overdemand ordering under RLDA, so that $\hat{C}_1^R \geq \hat{C}_2^R \geq \cdots \geq \hat{C}_N^R \geq \hat{C}_{N+1}^R$.

The idea will be to construct a set of cutoffs $\tilde{\mathbf{C}}^P$ directly from the permutation P and the cutoffs $\hat{\mathbf{C}}^R$, show that the cutoffs are in the correct order $\tilde{C}_1^P \geq \tilde{C}_2^P \geq \cdots \geq \tilde{C}_N^P \geq \tilde{C}_{N+1}^P$, and show that the cutoffs $\tilde{\mathbf{C}}^P$ and resulting assignment are market-clearing when school preferences are given by the amended scoring function with permutation P.

(1a) Definitions.

As in the proof of Theorem 1, let $\beta_{i,j} = \eta(\{\lambda \in \Lambda : \operatorname{argmax}_{\succ^\lambda} X_j = s_i\})$ be the measure of students who, when their set of affordable schools is X_j, will choose s_i. Let $E^\lambda(\mathbf{C})$ be the set of schools affordable for type λ in the first round under PLDA with any permutation, let $\hat{E}^\lambda(\hat{\mathbf{C}}^R)$ be the set of schools affordable for type λ in the second round under RLDA, and let $\hat{E}^\lambda(\hat{\mathbf{C}}^P)$ be the set of schools affordable for type λ in the second round under PLDA with permutation P.

Let $\gamma_i^R = \eta(\{\lambda \in \Lambda : \hat{E}^\lambda(\hat{\mathbf{C}}^R) = X_i\})$ be the fraction of students whose affordable set in the second round of RLDA is X_i, and let $\gamma_i^P = \eta(\{\lambda \in \Lambda : \hat{E}^\lambda(\hat{\mathbf{C}}^P) = X_i\})$ be the fraction of students whose affordable set in the second round of PLDA with permutation P is X_i.

Let $\hat{n}$ be the smallest index such that $s_{\hat{n}}$ does not reach capacity when it is not offered to all the students. In other words, $\hat{n}$ is the smallest index such that every student has school $s_{\hat{n}}$ in her affordable set under RLDA, i.e., $s_{\hat{n}} \in \hat{E}^\lambda(\hat{\mathbf{C}}^R)$. Since the local order condition holds for RLDA, we may equivalently express $\hat{n}$ in terms of cutoffs as the smallest index such that $(1-C_{\hat{n}})+(1-\hat{C}^R_{\hat{n}}) \geq 1$. Such an $\hat{n}$ always exists, since every student has the outside option s_{N+1} in her total affordable set.

(1b) Defining cutoffs for PLDA.

Let us define cutoffs $\tilde{\mathbf{C}}^P$ as follows. For $i \geq \hat{n}$ let $\tilde{C}^P_i = 0$. For each permutation P, define a function

$$f^P_i(x) = |\{l \,:\, l \geq C_i \text{ or } P(l) \geq x\}|$$

representing the proportion of students who have s_i in their (second-round) affordable set with first- and second-round cutoffs C_i, x under the amended scoring function with permutation P. Since P is measure-preserving, $f^P_i(x)$ is continuous and monotonically decreasing in x.

For $i < \hat{n}$, we inductively define $\tilde{C}^P_i$ to be the largest real smaller than $\tilde{C}^P_{i-1}$ satisfying

$$f^P_i(\tilde{C}^P_i) = f^R_i\left(\hat{C}^R_i\right) \tag{2}$$

(where we define $\tilde{C}^P_0 = 1$). Now $f^P_i(0) = 1 \geq f^R_i\left(\hat{C}^R_i\right)$, and

$$\begin{aligned} f^P_i\left(\tilde{C}^P_{i-1}\right) &= f^P_{i-1}\left(\tilde{C}^P_{i-1}\right) + |\{l \mid l \in [C_i, C_{i-1}) \text{ and } P(l) \geq \tilde{C}^P_{i-1}\}| \\ &\leq f^R_{i-1}\left(\hat{C}^R_{i-1}\right) + (C_{i-1} - C_i) \\ &= (1 - C_i) + (1 - \hat{C}^R_{i-1}) \\ &\leq f^R_i\left(\hat{C}^R_i\right) = f^P_i\left(\tilde{C}^P_i\right) \end{aligned}$$

where in the first equality we are using that $C_{i-1} \geq C_i$, the first inequality follows from the definition of $\tilde{C}^P_{i=1}$, and the last inequality holds since $\hat{C}^R_{i-1} \geq \hat{C}^R_i$.

It follows from the intermediate value theorem that the cutoffs $\tilde{\mathbf{C}}^P$ are well defined and satisfy $\tilde{C}^P_1 \geq \tilde{C}^P_2 \geq \cdots \geq \tilde{C}^P_N \geq \tilde{C}^P_{N+1}$.

(1c) The constructed cutoffs clear the market.

We show that the cutoffs $\tilde{\mathbf{C}}^P$ and resulting assignment (from letting students choose their favorite school out of those for which they meet the cutoff) are market-clearing when the second-round scores are given by $\hat{r}^\lambda_i = P(L(\lambda)) + n_i \mathbb{1}_{\{L(\lambda) \geq C_i\}} + p^\lambda_i \mathbb{1}_{\{L(\lambda) \geq C_i\}}$. We call the mechanism with this second-round assignment M^P.

The idea is that since the cutoffs $\tilde{C}^P_i$ are decreasing in the same order as C_i and $\hat{C}^R_i$, the (second-round) affordable sets are nested in the same order under both sets of second-round cutoffs.

It follows that aggregate student demand is uniquely specified by the proportion of students with each school in their affordable set, and we have defined these to be equal, $f_i^P\left(\tilde{C}_i^P\right) = f_i^R\left(\hat{C}_i^R\right)$. It follows that $\tilde{\mathbf{C}}^P$ are market-clearing and give the PLDA(P) cutoffs, and so PLDA(P) satisfies the local order condition (with the indices indexed in the same order as with RLDA). We make the affordable set argument explicit below.

Consider the proportion of lottery numbers giving a second-round affordable set X_i. Since $\hat{C}_1^R \geq \hat{C}_2^R \geq \cdots \geq \hat{C}_N^R$, under RLDA this is given by

$$\gamma_i^R = f_{i+1}^R\left(\hat{C}_{i+1}^R\right) - f_i^R\left(\hat{C}_i^R\right),$$

if $i < \hat{n}$ and by 0 if $i > \hat{n}$, where we define $f_0^P(x) = 1$ for all P and x. Similarly, since $\tilde{C}_1^P \geq \tilde{C}_2^P \geq \cdots \geq \tilde{C}_N^P$, under M^P this is given by

$$f_{i+1}^P\left(\tilde{C}_{i+1}^P\right) - f_i^P\left(\tilde{C}_i^P\right)$$

if $i < \hat{n}$, which is precisely γ_i^R, and by 0 if $i > \hat{n}$.

Hence, for all $i < \hat{n}$, the measure of students assigned to school s_i under both RLDA and M^P is $\sum_{j \leq i} \beta_{i,j}\gamma_j^R = q_i$, and for all $i \geq \hat{n}$, the measure of students assigned to school s_i is $\sum_{j \leq \hat{n}} \beta_{i,j}\gamma_j^R < q_i$. It follows that the cutoffs $\tilde{\mathbf{C}}^P$ are market-clearing when the second-round scores are given by $\hat{r}_i^\lambda = P(L(\lambda)) + n_i \mathbb{1}_{\{L(\lambda) \geq C_i\}} + p_i^\lambda \mathbb{1}_{\{L(\lambda) \geq C_i\}}$, and so $PLDA(\mathrm{P}) = M^P$ satisfies the local order condition.

(2) Some school has more than one priority group.

Now consider when schools have possibly more than one priority group. We show that if RLDA satisfies the local order condition, then PLDA with permutation P assigns the same number of students of each priority type to each school s_i as RLDA, and within each priority type assigns the same number of students of each preference type to each school as RLDA. We do this by first assuming that PLDA with permutation P assigns the same number of students of each priority type to each school s_i as RLDA, and showing that this gives consistent cutoffs.

We note that this proof uses very similar arguments to the proof of Theorem 6.

(2a) Defining little economies $\mathcal{E}_\pi$ for each priority type.

Fix a priority class π. Let q_π be a restricted capacity vector, where $q_{\pi,i}$ is the measure of students of priority class π assigned to school s_i under RLDA. Let Λ_π be the set of students λ such that

$p^\lambda = \pi$, and let η_π be the restriction of the distribution η to Λ_π. Let $\mathcal{E}_\pi$ denote the primitives $(S, q_\pi, \Lambda_\pi, \eta_\pi)$.

Let $\tilde{\mathbf{C}}_\pi^P$ be the second-round cutoffs of PLDA(P) on $\mathcal{E}_\pi$. By definition, $\hat{\mathbf{C}}_\pi^R$ are the second-round cutoffs of RLDA on $\mathcal{E}_\pi$. We show that the cutoffs $\tilde{\mathbf{C}}_\pi^P$ defined for the little economy are the same as the consistent second-round cutoffs $\hat{\mathbf{C}}_\pi^P$ for PLDA(P) run on the large economy $\mathcal{E}$, that is, $\tilde{\mathbf{C}}_\pi^P = \hat{\mathbf{C}}_\pi^P$.

(2b) Implications of RLDA satisfying the local order condition.

Since RLDA satisfies the local order condition for $\mathcal{E}$, RLDA also satisfies the local order condition for $\mathcal{E}_\pi$ for all π. It follows from (1) that the global order condition holds on each of the little economies $\mathcal{E}_\pi$. Hence by Theorem 1 all PLDA mechanisms produce type-equivalent assignments when run on $\mathcal{E}_\pi$. Moreover, as in the proof of Theorem 6, the global order condition on $\mathcal{E}_\pi$ also allows us to determine aggregate student demand in $\mathcal{E}_\pi$ from the proportions of students who have each school in their affordable set.

Finally, as in the proof of Theorem 6, we may assume that for each π and school s_i the cutoff $\tilde{C}_{\pi,i}^P$ is the minimal real satisfying

$$f_{\pi,i}^P\left(\tilde{C}_{\pi,i}^P\right) = f_{\pi,i}^R\left(\hat{C}_{\pi,i}^R\right)$$

where for each permutation P,

$$f_{\pi,i}^P(x) = |\{l : l \geq C_{\pi,i} \text{ or } P(l) \geq x\}|$$

is the proportion of students of priority class π who have school s_i in their affordable set if the first- and second-round cutoffs are $C_{\pi,i}$ and x respectively.

It follows that $\tilde{C}_{\pi,i}^P$ depends on π only via π_i, and does not depend on π_j for all $j \neq i$.

This is because $\tilde{C}_{\pi,i}^P$ is defined by $f_{\pi,i}^P(\cdot)$, $f_{\pi,i}^R(\cdot)$, and $\hat{C}_{\pi,i}^R$, which are in turn defined by $C_{\pi,i}$ and the permutations P and R. Moreover, $C_{\pi,i}$ depends on π only through π_i. Hence, if π, π' are two priority vectors such that $\pi_i = \pi_i'$, then $\tilde{C}_{\pi,i}^P = \tilde{C}_{\pi',i}^P$, and so the $\tilde{C}_{\pi,i}^P$ are consistent across priority types and define cutoffs $\tilde{C}_i^P$ that are independent of priority type.

(3) $\tilde{C}_i^P$ are the PLDA cutoffs.

Finally, we show that $\tilde{C}_i^P$ are market-clearing cutoffs. By (1), for each priority class π, the number of students assigned to each school s_i is the same under RLDA as under the demand induced by the cutoffs $\tilde{C}_i^P$, and we know that the RLDA cutoffs are market-clearing for $\mathcal{E}$.

Hence $\tilde{C}_i^P$ give the assignments for PLDA on $\mathcal{E}$, and since $\tilde{C}_i^P$ was defined individually for each priority class π for $\mathcal{E}_\pi$ it follows that PLDA(P) assigns the same measure of students of each priority type to each school s_i as RLDA.

□

C.5 Proof of Theorem 3

Proof of Theorem 3. We first note that with a single priority class, the first round corresponds to the random serial dictatorship (RSD) mechanism of Abdulkadiroglu and Sönmez (1998), where the (random) order of students is the single order of tie-breaking. Hence instead of referring to the first-round mechanism as DA-STB, we will sometimes refer to it as RSD.

Recall the cutoff characterization of the set of stable matchings for given student preferences and responsive school preferences (encoded by student scores $r_i^\lambda = p_i^\lambda + L(\lambda)$) Azevedo and Leshno (2016). Namely, if $\mathbf{C} \in \mathbb{R}_+^{\mathbf{N}}$ is a vector of cutoffs, let the assignment μ defined by $\mathbf{C}$ be given by assigning each student of type λ to her favorite school among those where her score weakly exceeds the cutoff, $\mu(\lambda) = \max_{\succ^\lambda}(\{s_i \in S : r_i^\lambda \geq C_i\} \cup \{s_{N+1}\})$. The cutoffs $\mathbf{C}$ are market-clearing if under the assignment μ defined by $\mathbf{C}$, every school with a positive cutoff is exactly at capacity, $\eta(\mu(s_i)) \leq q_i$ for all $s_i \in S$, with equality if $C_i > 0$. The set of all stable matchings is precisely given by the set of assignments defined by market-clearing vectors (Azevedo and Leshno, 2016).

Under PLDA(P), a student of type λ has a second-round score $\hat{r}_i^\lambda = P(L(\lambda)) + \mathbb{1}_{\{L(\lambda) \geq C_i\}}$ at school s_i for each school $s_i \in S \cup \{s_{N+1}\}$ (assuming that scores are modified to give guarantees to students who had a school in their first-round affordable set, instead of just students assigned to the school in the first round). In a slight abuse of notation, we will sometimes let $\hat{\mathbf{C}}^P$ refer to the second-round cutoffs from some fixed PLDA(P) (not necessarily corresponding to the student-optimal stable matching given by PLDA).

The proof that any PLDA satisfies the axioms essentially follows from Proposition 1. We note that averaging follows from the continuum model, which preserves the relative proportion of students with different reported types under random lotteries and permutations of random lotteries. Hence it suffices to show that any mechanism M satisfying the axioms is a PLDA.

We will show that the reassignment produced by M is type-equivalent to the reassignment produced by some PLDA. If we assume that, conditional on their reports, students' assignments under M are uncorrelated, we are able to explicitly construct a PLDA that provides the same joint distribution over assignments and reassignments as M. We provide a sketch of the proof before fleshing out the details.

Fix a distribution of student types ζ. Since the first round of our mechanism M is DA-STB and M is anonymous, this gives a distribution η of students that is the same (up to relabeling of students) at the end of the first round. For a fixed labeling of students, it also gives a distribution over first-round assignments μ and a distribution over second round assignments $\tilde{\mu}$.

We first invoke averaging to assume that all ensuing constructions of aggregate cutoffs and measures of students assigned to pairs of schools in the two rounds are deterministic. Specifically, since the first-round assignment μ is given by STB, and the mechanism satisfies the averaging axiom, we may assume that each pathwise realization of the mechanism gives type-equivalent (two-round) assignments. Hence, for the majority of the proof we perform our constructions of aggregate cutoffs and measures of students pathwise, and assume that any realization of the lottery numbers produces the same cutoffs and measures of students. (In particular, the quantities $\hat{C}_i, \rho_{i,j}, \gamma_{i,j}$ that we will later define will be the same across all realizations.)

Outline of Proof. We use constrained Pareto efficiency to construct a first-round overdemand ordering $s_1, s_2, \ldots, s_N, s_{N+1}$ and a permutation σ giving the second-round overdemand ordering $s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(N+1)}$, as in (Ashlagi and Shi, 2014), where school s comes before s' in an ordering for the first (second) round if there exists a non-zero measure of students who prefer school s to s' in the first (second) round but who are assigned to s' in the first (second) round. (In the case of the second-round ordering, we require that these students' second-round assignments s' not be the same as their first-round guarantees.) The existence of these orderings follows from the facts that the first-round mechanism, DA with a single priority class and uniform-at-random single tie-breaking, is Pareto efficient, that the two-round mechanism is constrained Pareto efficient. We let $X_i = \{s_i, s_{i+1}, \ldots, s_{N+1}\}$ denote the set of schools after s_i in the first-round overdemand ordering, and let $\tilde{X}_i = \{s_{\sigma(i)}, s_{\sigma(i+1)}, \ldots, s_{\sigma(N+1)}\}$ denote the set of schools after $s_{\sigma(i)}$ in the second-round overdemand ordering.

We next note that instead of assignments μ and $\hat{\mu}$, we can think of giving students first- and second-round affordable sets $E(\lambda), \hat{E}(\lambda)$ so that μ and $\hat{\mu}$ are given by letting each student choose her favorite school in her affordable set for that round. We use weak two-round strategy-proofness and anonymity to show that two students of different types face the same joint distribution over first-and second-round affordable sets. This allows us to construct the permutation P by constructing proportions $\gamma_{i,j}$ of students whose first-round affordable set was X_i and whose (second-round) affordable set was $\tilde{X}_j$. This is the most technical step in the proof, and so we separate it into several steps. The crux of the analysis is the fact that for any school s and set $S' \not\ni s$ of schools, two students with top choices S' who are assigned to a school they weakly prefer to s the first round have the same conditional probability of being assigned to a school in S' in the second round.[5] We term this the "prefix property" and prove it in Lemma 3.

Finally, we construct the lottery L and verify that if second-round scores are given by first prioritizing all guaranteed students over non-guaranteed students and subsequently breaking ties according to the permuted lottery $P \circ L$, then PLDA(P) gives every student the same pair of first- and second- round assignments as M.

Formal Proof. We now present the formal proof. Since we are assuming that the considered mechanism M is weakly strategy-proof, we assume that students report truthfully and so we consider preferences instead of reported preferences. We will explicitly specify when we are considering the possible outcomes from a single student misreporting.

(2a) Definitions

Let the schools be numbered $s_1, s_2, \ldots, s_N$ such that $C_i \geq C_{i+1}$ for all i. The intuition is that this is the order in which they reach capacity in the first round. We observe that all reassignments are index-decreasing. That is, for all s, s', if there exists a non-zero measure of students who are assigned to s in the first round and to s' in the second round, and $s' \neq s_{N+1}$, then $s = s_i$ and $s' = s_j$ for some $i \geq j$. This follows since the mechanism respects guarantees, student preferences are consistent, and the schools are indexed in order of increasing first-round affordability. Throughout this section we will denote the outside option s_{N+1} either by s_0 or $\emptyset$, to make it more evident that indices are decreasing.

Next, we define a permutation σ on the schools. We think of this as giving a second-round overdemand (or inverse affordability) ordering, where in the second round the schools fill in the order $s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(N)}$. We will eventually show that M gives the same outcome as a PLDA with cutoffs that are ordered $\hat{C}_{\sigma(1)} \geq \hat{C}_{\sigma(2)} \geq \cdots \geq \hat{C}_{\sigma(N)}$. We require that σ satisfies the following property. For all s, s', if there exists a non-zero measure of students with consistent preferences who have second-round preference reports $\succ$ such that $s \succ s'$, and who are not assigned to s' in

[5]The formal statement also takes into account how demanded the schools they weakly prefer to s are, and is given in terms of student types who were assigned to s, and lottery numbers.

the first round, but are assigned to s' in the second round, then $s = s_{\sigma(i)}$ and $s' = s_{\sigma(j)}$ for some $i < j$. We assume that σ is the unique permutation satisfying this property that is maximally order-preserving. That is, for all pairs of schools s_i, s_j for which no non-zero measure of students of the above type exists, $\sigma(i) < \sigma(j)$ iff $i < j$. We also define $\sigma(N+1) = N+1$. An ordering σ with the required properties exists since the mechanism is constrained Pareto efficient. In particular, if there is a cycle of schools $s_{i_1}, s_{i_2}, \ldots, s_{i_m}$ where for each j there is a set of students Λ_j with non-zero measure who prefer $s_{i_{j+1}}$ to their second-round assignment s_{i_j} and who are not assigned to s_{i_j} in the first round, then $\hat{p}^{\lambda_j}_{s_{i_j}} = p^{\lambda_j}_{s_{i_j}}$ for each $\lambda_j \in \Lambda_j$, and so there is a Pareto-improving cycle that respects second-round priorities.

Let S' be a set of schools, and let $\succ$ be a preference ordering over all schools. We say that S' is a *prefix* of $\succ$ if $s' \succ s$ for all $s' \in S', s \notin S'$. For a set of schools S', let $i(S') = \max\{j : s_j \in S'\}$ be the maximum index of a school in S'. We may think of $i(S')$ as the index of the most affordable school in S' in the first round. For a student type $\theta = (\succ, \hat{\succ})$, an interval $I \subseteq [0,1]$, and a set of schools S', let $\rho^\theta(I, S')$ be the proportion of students with type θ who, under the mechanism M, have a first-round lottery in the interval I and are assigned to a school in S' in the second round. When $S' = \{s'\}$ we will sometimes write $\rho^\theta(I, s')$ instead of[6] $\rho^\theta(I, \{s'\})$. In this section, for brevity, when defining preferences $\succ$ we will sometimes write $\succ$: $[s_{i_1}, s_{i_2}, \ldots, s_{i_k}]$ instead of $s_{i_1} \succ s_{i_2} \succ \cdots \succ s_{i_k}$.

(2b) Constructing the permutation P.

We now construct the permutation P as follows. For all pairs of indices i, j, we define a scalar $\gamma_{i,j}$, which we will show can be thought of as the proportion of students (of any type) whose first-round affordable set is X_i and whose second-round affordable set is $\tilde{X}_j$.

Now, for all pairs of indices i, j such that $\sigma(j) < i$, we define student preferences $\theta_{i,j} = (\succ_{i,j}, \hat{\succ}_{i,j})$ such that

$$\succ_{i,j}: [s_{\sigma(j)}, s_{i-1}, s_i, s_{N+1}] \quad \text{and} \quad \hat{\succ}_{i,j} : [s_{\sigma(j)}, s_{N+1}],$$

with all other schools unacceptable. (We remark that in the case where $\sigma(j) = i-1$, the first two schools in this preference ordering coincide.) We note that the full-support assumption implies that there is a positive measure of such students. Let $\rho_{i,j}$ be the proportion of students of type $\theta_{i,j}$ whose first-round assignment is s_i and whose second-round assignment is school $s_{\sigma(j)}$. Intuitively, $\rho_{i,j}$ is the proportion of students who can deduce that their lottery number is in the interval $[C_i, C_{i-1}]$,

[6] Here we are assuming that this proportion is the same for every realization of the first round of M. This requires non-atomicity and anonymity.

and whose second-round affordable set contains $\tilde{X}_j$.

For a fixed index i, we define $\gamma_{i,j}$ for $j = 1, 2, \ldots, N$ to be the unique solutions to the following n equations:

$$\gamma_{i,j} = 0 \quad \text{for all } j \text{ such that } \sigma(j) \geq i$$
$$\gamma_{i,1} + \cdots + \gamma_{i,j} = \rho_{i,j} \quad \text{for all } j \text{ such that } \sigma(j) < i.$$

Note that by this definition it holds that $\gamma_{1,j} = 0$ for all j. We may intuitively think of $\gamma_{i,j}$ as the proportion of students of type $\theta_{i,j}$ whose first-round lottery is in $[C_i, C_{i-1}]$ and whose second-round affordable set contains $s_{\sigma(j)}$ but not $s_{\sigma(j-1)}$. (This is not quite the case, as we let $\gamma_{i,j} = 0$ for all j such that $\sigma(j) \geq i$. More precisely, if $\sigma(j) < i$ then $\gamma_{i,j}$ is the proportion of students of type $\theta_{i,j}$ whose first-round lottery is in $[C_i, C_{i-1}]$ and whose second-round affordable set contains $s_{\sigma(j)}$, but not $s_{\sigma(j')}$, where $j' = \max\{j'' : \sigma(j'') < i\}$.) Note that if $\sigma(j) \geq i$ then school $s_{\sigma(j)}$ will be in the first-round affordable set for all students whose first-round lottery is in $[C_i, C_{i-1}]$, and we define $\gamma_{i,j} = 0$ and keep track of these students separately.

We also define $\gamma_{i,N+1}$ to be

$$\gamma_{i,N+1} = C_{i-1} - C_i - \sum_{j=1}^{n} \gamma_{i,j}.$$

Since transfers are index-decreasing, we may intuitively think of $\gamma_{i,N+1}$ as the proportion of students of type $\theta_{i,j}$ assigned to school s_i in the first round whose only available school in the second round comes from their first-round guarantee.

We define the lottery P from $\gamma_{i,j}$ as follows. We break the interval $[0, 1]$ into $(N+1)^2$ intervals, $\tilde{I}_{i,j}$, where the interval $\tilde{I}_{i,j}$ has length $\gamma_{i,j}$, and the intervals are ordered in decreasing order of the first index[7] i,

$$\tilde{I}_{N+1,N+1}, \tilde{I}_{N+1,N}, \ldots, \tilde{I}_{1,2}, \tilde{I}_{1,1}.$$

The permutation P maps the intervals back into $[0, 1]$ in decreasing order of the second index[8] j,

$$P(\tilde{I}_{N+1,N+1}), P(\tilde{I}_{N,N+1}), \ldots P(\tilde{I}_{2,1}), P(\tilde{I}_{1,1}).$$

In Figure 5, we show an example with two schools.

[7] Specifically, let $\tilde{I}_{i,j} = [C_{i-1} - \sum_{j' \leq j} \gamma_{i,j'}, C_{i-1} - \sum_{j' < j} \gamma_{i,j'}]$.

[8] Specifically, let $\hat{C}_{\sigma(j)} = 1 - \sum_{i',j' : j' \leq j} \gamma_{i',j'}$, and let $P(\tilde{I}_{i,j}) = [\hat{C}_{\sigma(j-1)} - \sum_{i' \leq i} \gamma_{i',j}, \hat{C}_{\sigma(j-1)} - \sum_{i' < i} \gamma_{i',j}]$.

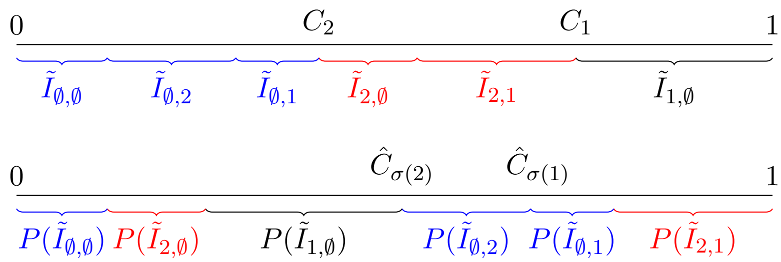

Figure 1: Constructing the permutation P for $n = 2$ schools, where σ is the identity permutation. The intervals $\tilde{I}_{i,j}$ for $i \leq \sigma(j) = j < N + 1$ are empty by definition, as all transfers are index-decreasing.

We note that $\sum_{j=1}^{N+1} \gamma_{i,j} = C_{i-1} - C_i$, which is the proportion of students whose first-round affordable set is X_i. We may interpret $\gamma_{i,j}$ to be the proportion of students who can deduce that their lottery number is in the interval $[C_i, C_{i-1}]$, and whose second-round affordable set is $\tilde{X}_j$, and so $\sum_{i=1}^{N+1} \gamma_{i,j}$ is the proportion of students whose second-round affordable set is $\tilde{X}_j$. We remark that there may be multiple values of i, j for which $\gamma_{i,j} = 0$ (i.e. there are no students whose first-round affordable set is X_i and second-round affordable set is X_j), but that this does not affect our ability to assign students to all possible pairs of schools that are consistent with consistent preferences and the first- and second-round overdemand orderings. For example $\gamma_{1,j} = 0$ for all j, but any student whose first-round affordable set is X_1 is assigned to her top choice school in both rounds, and hence her second-round affordable set is inconsequential.

We show that there exists a PLDA mechanism with permutation P, where the students with first-round scores in $\tilde{I}_{i,j}$ are precisely the students with a first-round affordable set X_i and a second-round affordable set $\tilde{X}_j$, and that this PLDA mechanism gives the same joint distribution over first- and second-round assignments as M. To do this, we first show that this distribution of first- and second-round affordable sets gives rise to the correct joint first- and second-round assignments over all students. We then use anonymity to construct L in such a way as to have the correct first- and second-round assignment joint distributions for each student. Finally, we verify that these second-round affordable sets give the student-optimal stable matching under the second round school preferences given by P.

(2c) Equivalence of the joint distribution of assignments given by affordable sets and M.

Fix student preferences $\theta = (\succ, \hat{\succ})$. We show that if we let $\gamma_{i,j}$ be the proportion of students with preferences θ who have first- and second-round affordables X_i and $\tilde{X}_j$ respectively, then we obtain the same joint distribution over assignments in the first and second rounds for students with preferences θ as under mechanism M. In doing so, we will use the following "prefix lemma".

The "prefix lemma" states that for every set of schools S', there exist certain intervals of the form $I_i^j = [C_i, C_j]$ such that for any two student types whose top set of acceptable schools under second-round preference reports is S', the proportion of students with lotteries in I_i^j who are upgraded to a school in S' in the second round is the same for each type.

We define a *prefix* of preferences $\succ$ to be a set of schools S' that is a top set of acceptable schools under $\succ$; that is, for all $s' \in S'$ and $s \notin S'$, it holds that $s' \succ s$.

Lemma 3. *[Prefix Property] Let $s = s_j$ be a school, and let $S' \not\ni s$ be a set of schools such that $i(S') < j$. Let $\theta = (\succ, \hat{\succ})$ and $\theta' = (\succ', \hat{\succ}')$ be consistent preferences such that S' is a prefix of $\succ, \hat{\succ}$ and some students with preferences θ are assigned to school s in the first round, and similarly S' is a prefix of $\succ', \hat{\succ}'$ and some students with preferences θ' are assigned to school s in the first round.*

Then

$$\rho^{\theta}([C_j, C_{i(S')}], S') = \rho^{\theta'}([C_j, C_{i(S')}], S').$$

That is, the proportion of students of type θ whose first-round lotteries are in the interval $[C_j, C_{i(S')}]$ and who are assigned to a school in S' in the second round is the same as the proportion of students of type θ' whose first-round lotteries are in the interval $[C_j, C_{i(S')}]$ and who are assigned to a school in S' in the second round.

Sketch of proof of Lemma 3. The idea of the proof is to use weak strategy-proofness and first-order stochastic dominance to show that the probabilities of being assigned to S' (conditional on certain first-round assignments) are the same for students of type θ or θ'. We then invoke anonymity to argue that proportions of types of students assigned to a certain school are given by the conditional probabilities of individual students being assigned to that school. We present the full proof at the end of Section 3.1. $\square$

We now show that the mechanism M and the affordable set distribution $\gamma_{i,j}$ produce the same joint distribution of assignments.

(2c.i.) Students with two acceptable schools.

To give a bit of the flavor of the proof, we first consider student preferences θ of the form $\succ$: $[s, s', s_{N+1}]$ and $\hat{\succ}$: $[s, s_{N+1}]$, where all other schools are unacceptable. We let k, l be the indices such that $s = s_k$ and $s' = s_l$.

There are five ordered pairs of schools that students of this type can be assigned to in the two rounds. Namely, if we let (s, s') denote assignment to s in the first round and to s' in the second round, then the ordered pairs are (s, s), (s', s), (s', s_{N+1}), (s_{N+1}, s), and (s_{N+1}, s_{N+1}). Since the proportion of students with each first-round assignment is fixed, it suffices to show that the mechanism M and the mechanism that assigns first- and second-round affordable set distributions according to $\gamma_{i,j}$ produce the same proportion of students assigned to (s', s) and the same proportion of students assigned to (s_{N+1}, s).

Let $I_l^k = [C_l, C_k]$, and let $I_{N+1}^{\max\{k,l\}} = [0, C_{\max\{k,l\}}]$. The proportions of students with preferences θ who are assigned to (s', s) and (s_{N+1}, s) under M are given by $\rho^\theta([C_l, C_k], s)$ and $\rho^\theta([0, C_{\max\{k,l\}}], s)$ respectively. We want to show that this is the same as the proportion of students with preferences θ who are assigned to (s', s) and (s_{N+1}, s) respectively when first- and second-round affordable sets are given by the affordable set distribution $\gamma_{i,j}$. We remark that when $k > l$ this holds vacuously, since all the terms are 0. Hence, since for any school s the proportion of students with preferences θ who are assigned to s in the first round does not depend on θ, it suffices to consider the case where $k < l$.

Let $\theta' = (\succ', \hat{\succ}')$ be the preferences given by $\succ'$: $[s = s_k, s_{k+1}, \ldots, s_{l-1}, s' = s_l, s_{N+1}]$ and $\hat{\succ}' : [s = s_k, s_{N+1}]$, where only the schools with indices between k and l are acceptable in the first round, only $s = s_k$ is acceptable in the second round, and all other schools are unacceptable.

For all pairs of indices i, j such that $j < i$, let $\theta'_{i,j} = (\succ_{i,j}, \hat{\succ}_{i,j})$ be the student preferences such that $\succ_{i,j}$: $[s_j, s_{i-1}, s_i, s_{N+1}]$ and $\hat{\succ}_{i,j} : [s_j, s_{N+1}]$, with all other schools unacceptable. (In the case where $i = j + 1$, we let the first two schools under the preference ordering $\succ_{i,j}$ coincide.) We note that $\theta'_{i,j} = \theta_{i,\sigma^{-1}(j)}$, where $\theta_{i,j}$ was defined in (2b), and that for $i > \sigma(j)$ we previously defined $\rho_{i,j} = \sum_{l \leq j} \gamma_{i,l}$ to be the proportion of students of type $\theta_{i,j}$ whose first-round assignment is s_i and whose second-round assignment is school s_j.

The proportion of students with preferences θ who are assigned to (s', s) under M is given by

$$\begin{aligned}
\rho^{\theta}([C_l, C_k], s) &= \rho^{\theta'}([C_l, C_k], s) \text{ (by the prefix property (Lemma 3))} \\
&= \sum_{k<i\le l} \rho^{\theta'}([C_i, C_{i-1}], s) \\
&= \sum_{k<i\le l} \rho^{\theta'_{i,k}}([C_i, C_{i-1}], s) \\
&\quad \text{(since the second-round assignment does not depend on the first-round report)} \\
&= \sum_{k<i\le l} \rho_{i,\sigma^{-1}(k)} \text{ (by the definition of } \rho_{i,\sigma^{-1}(k)}) \\
&= \sum_{k<i\le l} \sum_{j\le\sigma^{-1}(k)} \gamma_{i,j} \text{ (by the definition of } \gamma_{i,j}),
\end{aligned}$$

which is precisely the proportion of students with preferences θ who are assigned to (s', s) if the first- and second-round affordable sets are given by $\gamma_{i,j}$.

Similarly, let $\theta'' = (\succ'', \hat{\succ}'')$ be the preferences given by $\succ''$: $[s = s_k, s' = s_l, s_{l+1}, \ldots, s_N, s_{N+1}]$ and $\hat{\succ}''$: $[s = s_k, s_{N+1}]$, where only $s = s_k$ and the schools with indices greater than l are acceptable in the first round, only $s = s_k$ is acceptable in the second round, and all other schools are unacceptable. Then the proportion of students with preferences θ who are assigned to (s_{N+1}, s) under M is given by

$$\begin{aligned}
\rho^{\theta}([0, C_l], s) &= \rho^{\theta''}([0, C_l], s) \text{ (by the prefix property (Lemma 3))} \\
&= \sum_{l<i\le N} \rho^{\theta''}([C_i, C_{i-1}], s) \\
&= \sum_{l<i\le N} \rho^{\theta'_{i,k}}([C_i, C_{i-1}], s) \\
&\quad \text{(since the second-round assignment does not depend on the first-round report)} \\
&= \sum_{l<i\le N} \rho_{i,\sigma^{-1}(k)} \text{ (by the definition of } \rho_{i,\sigma^{-1}(k)}) \\
&= \sum_{i,j:l<i\le N, j\le\sigma^{-1}(k)} \gamma_{i,j} \text{ (by the definition of } \gamma_{i,j}),
\end{aligned}$$

which is precisely the proportion of students with preferences θ who are assigned to (s_{N+1}, s) if the first- and second-round affordable sets are given by $\gamma_{i,j}$.

(2c.ii.) Students with general preferences.

We now consider general (consistent) student preferences θ of the form $(\succ, \hat{\succ})$, where

$$\succ: [s_{i_1}, s_{i_2}, \ldots, s_{i_k}, s_{N+1}] \text{ and } \hat{\succ}: [s_{i_1}, s_{i_2}, \ldots, s_{i_l}, s_{N+1}],$$

for some $k > l$ and where all other schools are unacceptable. We wish to show that for every pair of schools $s, s' \in \{s_{i_1}, s_{i_2}, \ldots, s_{i_k}, s_{N+1}\}$, the mechanism M and the mechanism that assigns first- and second-round affordable set distributions according to $\gamma_{i,j}$ produce the same proportion of students assigned to (s, s'). It suffices to show that for every prefix S' of the preferences $\hat{\succ}$ and every school $s \in \{s_{i_2}, \ldots, s_{i_k}, s_{N+1}\}$, the mechanism M and the mechanism that assigns first- and second-round affordable set distributions according to $\gamma_{i,j}$ produce the same proportion of students assigned to s in the first round and some school in S' in the second round. We say that the students are assigned to (s, S').

Fix a prefix S' of $\hat{\succ}$ and a school $s = s_{i_j}, 1 < j \leq k$. Let $m \leq k$ be such that $S' = \{s_{i_1}, s_{i_2}, \ldots, s_{i_m}\}$. If $j \leq m$ then $s \in S'$, and so in any mechanism that respects guarantees, the proportion of students assigned to (s, S') is the same as the proportion of students assigned to s in the first round.

Recall that $i(S')$ is the largest index of a school in S', i.e. if $i(S') = max\{i' : s_{i'} \in S'\}$. (Note that this is not necessarily i_m, the index of the school in S' that is least preferred by a student of type θ.) If $j > m$ and $i_j \leq i(S')$, then in the first round, whenever the school i_j is available in the first round, so is the preferred school $i(S')$; thus, for any school s', the proportion of students assigned to s in the first round is 0. It follows that in any mechanism that respects guarantees, the proportion of students assigned to (s, S') is 0.

From here on, we may assume that $j > m$ (i.e., $s \notin S'$) and $i_j > i(S')$. Since $i_j > i(S')$, the proportion of students with preferences θ who are assigned to (s, S') under M is given by $\rho^{\theta}([C_{i_j}, C_{i(S')}], S')$. Let $i(\sigma(S'))$ be the index i such that $s_i \in S'$ and $\sigma^{-1}(i)$ is maximal, that is, the index of the school in S' that is most affordable in the second round.

Let $\theta' = (\succ', \hat{\succ}')$ be the preferences given by

$$\succ': [s_{i(\sigma(S'))}, s', s_{i(S')+1}, s_{i(S')+2}, \cdots, s_{i_j - 1}, s_{i_j}, s_{N+1}] \text{ and } \hat{\succ}': [s_{i(\sigma(S'))}, s', s_{N+1}]$$

for all $s' \in S' \setminus \{s_{i(\sigma(S'))}\}$. It may be helpful to think of this as all preferences of the form

$$\succ': [S', s_{i(S')+1}, s_{i(S')+2}, \cdots, s_{i_j-1}, s_{i_j}, s_{N+1}] \text{ and } \hat{\succ}' : [S', s_{N+1}],$$

where the school $s_{i(\sigma(S'))}$ comes first and otherwise the schools in S' are ordered arbitrarily, and where all schools between $s_{i(S')}$ and s_{i_j} are acceptable in the same order as first round overdemand.

We remark that only the schools in S' and the schools with indices between $i(S')$ and i_j are acceptable in the first round, only the schools in S' are acceptable in the second round, and all other schools are unacceptable. Since $j > m$, $i_j > i(S')$, and the preferences θ are consistent, the preferences θ' are well defined. Let $\theta'' = (\succ'', \hat{\succ}'')$ be the preferences given by $\succ''=\succ'$ and $\hat{\succ}'' : [s_{i(\sigma(S'))}, s_{N+1}]$.

Recall that for all $i > i(\sigma(S'))$, $\theta'_{i,i(\sigma(S'))} = (\succ_{i,i(\sigma(S'))}, \hat{\succ}_{i,i(\sigma(S'))})$ are the student preferences such that

$$\succ_{i,i(\sigma(S'))}: [s_{i(\sigma(S'))}, s_{i-1}, s_i, s_{N+1}] \text{ and } \hat{\succ}_{i,i(\sigma(S'))} : [s_{i(\sigma(S'))}, s_{N+1}],$$

with all other schools unacceptable. Additionally, recall that $\rho_{i,\sigma^{-1}(i(\sigma(S')))}$ is the proportion of students of type $\theta_{i,i(\sigma(S'))}$ whose first-round assignment is s_i and whose second-round assignment is school $s_{i(\sigma(S'))}$.

Let $\hat{S} = \{s_{i_1}, s_{i_2}, \ldots, s_{i_{j-1}}\}$, and let $i(\hat{S})$ be the index i such that $i \in \hat{S}$ and $\sigma^{-1}(i)$ is maximal, that is, the index of the school preferable to s under $\succ$ that is most affordable in the second round.

Then the proportion of students with preferences θ who are assigned to (s, S') under M is given by $\rho^{\theta}([C_{i_j}, C_{i(\hat{S})}], S')$, where

$$\begin{aligned}
\rho^{\theta}([C_{i_j}, C_{i(\hat{S})}], S') &= \rho^{\theta'}([C_{i_j}, C_{i(\hat{S})}], S') \text{ (by the prefix property (Lemma 3) with prefix } S'\text{)} \\
&= \sum_{i(\hat{S})<i\leq i_j} \rho^{\theta'}([C_i, C_{i-1}], S') \\
&= \sum_{i(\hat{S})<i\leq i_j} \rho^{\theta'}([C_i, C_{i-1}], s_{i(\sigma(S'))}) \\
&\qquad \text{(by the definition of the second-round overdemand ordering)} \\
&= \sum_{i(\hat{S})<i\leq i_j} \rho^{\theta''}([C_i, C_{i-1}], s_{i(\sigma(S'))}) \text{ (by the prefix property with prefix } \{s_{i(\sigma(S'))}\}\text{)} \\
&= \sum_{i(\hat{S})<i\leq i_j} \rho^{\theta'_{i,i(\sigma(S'))}}([C_i, C_{i-1}], s_{i(\sigma(S'))}) \\
&\quad \text{(since the second-round assignment does not depend on the first-round report)} \\
&= \sum_{i(\hat{S})<i\leq i_j} \rho_{i,\sigma^{-1}(i(\sigma(S')))} \text{ (by the definition of } \rho_{i,\sigma^{-1}(i(\sigma(S')))}\text{)} \\
&= \sum_{i(\hat{S})<i\leq i_j} \sum_{j'\leq\sigma^{-1}(i(\sigma(S')))} \gamma_{i,j'} \text{ (by the definition of } \gamma_{i,j'}\text{)},
\end{aligned}$$

which is precisely the proportion of students with preferences θ who are assigned to (s, S') if the first- and second-round affordable sets are given by $\gamma_{i,j'}$. Note that all $\theta'_{i,i(\sigma(S'))}$ and $\rho_{i,\sigma^{-1}(i(\sigma(S')))}$ in the summation are well-defined, since the sum is over indices satisfying $i > i\left(\hat{S}\right)$, and since $j > m$ it follows that $\hat{S} \supseteq S'$ and hence $i > i\left(\hat{S}\right) \geq i\left(\sigma\left(S'\right)\right)$.

(2d) Constructing the lottery L.

Fix a student λ who reports first- and second-round preferences $\theta = (\succ, \hat{\succ})$. Suppose that λ is assigned to schools (s_i, s_j) in the first and second rounds respectively. We first characterize all first- and second-round budget sets consistent with the overdemand orderings that could have led to this assignment. Let $\underline{i}$ be the smallest index i' such that $\max_{\succ} X_{i'} = s_i$, let $\underline{j}$ be the smallest index j' such that $\max_{\hat{\succ}} \tilde{X}_{j'} \cup \{s_i\} = s_j$, and let $\overline{j}$ be the largest index j' such that $\max_{\hat{\succ}} \tilde{X}_{j'} \cup \{s_i\} = s_j$. Then the set of first- and second-round budget sets that student λ could have been assigned by the mechanism is given by $\{X_{i'}, X_{j'} \cup \{s_i\} : \underline{i} \leq i' \leq i, \underline{j} \leq j' \leq \overline{j}\}$. (We remark that the asymmetry in these definitions is due to the existence of the first-round guarantee in the second-round budget sets.)

Conditional on λ being assigned to schools (s_i, s_j) in the first and second rounds respectively, we assign a lottery number $L(\lambda)$ to λ distributed uniformly over the union of intervals

$\cup_{i',j':\underline{i}\le i'\le i,\underline{j}\le j'\le\bar{j}}\tilde{I}_{i',j'}$,

$$(\ L(\lambda) \mid (\mu(\lambda),\tilde{\mu}(\lambda)) = (s_i,s_j)) \ \sim \ \text{Unif}\left(\cup_{i',j':\underline{i}\le i'\le i,\underline{j}\le j'\le\bar{j}}\tilde{I}_{i',j'}\right),$$

independent of all other students' assignments.

We show that this is consistent with the first round of the mechanism being RSD. We have shown in *(1)* that if for each pair of reported preferences $\theta = (\succ,\hat{\succ}) \in \Theta$, a uniform proportion $\gamma_{i',j'}$ of students with reported preferences θ are given first- and second-round budget sets $X_{i'}, \{s^\theta\} \cup \tilde{X}_{j'}$ (where $s^\theta = \max_\succ X_i$ is the first-round assignment of such students), we obtain the same distribution of assignments as M. Since M is anonymous and satisfies the averaging axiom, and since $|\tilde{I}_{i',j'}| = \gamma_{i',j'}$, it follows that each student's first-round lottery number is distributed as Unif$[0,1]$.

Given the constructed lottery L, we construct the second-round cutoffs $\hat{C}_i$ for the PLDA and verify that the assignment $\tilde{\mu}$ is feasible and stable with respect to the schools' second-round preferences, as defined by $P \circ L$ and the guarantee structure. Specifically, in PLDA, each student with a first-round score l and a first-round assignment s has a second-round score $\hat{r}_i = P(l) + \mathbb{1}(s = s_i)$ at each school $s_i \in S$, and students are assigned to their favorite school s_i at which their second-round score exceeds the school's second-round cutoff, $\hat{r}_i \ge \hat{C}_i$ (or to the outside option s_{N+1}).

Recall that the schools are indexed so that $C_1 \ge C_2 \ge \cdots \ge C_{N+1}$, and that the permutation σ is chosen so that the second-round overdemand ordering is given by $s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(N+1)} = s_{N+1}$, and so it should follow that the second-round cutoffs $\hat{C}_i$ satisfy $\hat{C}_{\sigma(1)} \ge \hat{C}_{\sigma(2)} \ge \cdots \ge \hat{C}_{\sigma(N+1)}$.

By the characterization of stable assignments given by Azevedo and Leshno (2016), it suffices to show that if each student with a first-round assignment s and second-round lottery number in $[\hat{C}_{\sigma^{-1}(i)}, \hat{C}_{\sigma^{-1}(i-1)}]$ is assigned to her favorite school in $\{s\} \cup \tilde{X}_i$, where we define $\tilde{X}_i = \{s_{\sigma(i)}, s_{\sigma(i+1)}, \ldots, s_{\sigma(N+1)}\}$, then the resulting assignment $\hat{\mu}$ is equal to the second-round assignment $\tilde{\mu}$ of our mechanism M, and satisfies that $\eta(\hat{\mu}^{-1}(s_i)) \le q_i$ for any school s_i, and $\eta(\hat{\mu}^{-1}(s_i)) = q_i$ if $\hat{C}_i > 0$.

For fixed i, j, let $\hat{C}_{\sigma(j)} = 1 - \sum_{i',j':j'\le j} \gamma_{i',j'}$ and let $\hat{C}_{i,\sigma(j)} = \hat{C}_{\sigma(j-1)} - \sum_{i'\le i} \gamma_{i',j}$. (We remark that since $\gamma_{i,j}$ refers to the i-th school to fill in the first round, s_i, and the j-th school to fill in the second round, $s_{\sigma(j)}$, the $\hat{C}$ are indexed slightly differently than $\gamma_{i,j}$ is.)

We use the averaging assumption and the equivalence of assignment probabilities that we have shown in (1) to conclude that if $\hat{\mu}$ is the assignment given by running DA with round scores $\hat{r}$ and cutoffs $\hat{\mathbf{C}}$, then $\tilde{\mu} = \hat{\mu}$.

This is fairly evident, but we also show it explicitly below. Specifically, consider a student $\lambda \in \Lambda$ with a first-round lottery number $L(\lambda)$ and reported preferences $\theta = (\succ, \hat{\succ})$. Let i, j be such that $L(\lambda) \in \cup_{i',j':\underline{i} \leq i' \leq i, \underline{j} \leq j' \leq \overline{j}} \tilde{I}_{i',j'}$, where $\underline{i}$ is the smallest index i' such that $\max_{\succ} X_{i'} = s_i$, $\underline{j}$ is the smallest index j' such that $\max_{\hat{\succ}} \tilde{X}_{j'} \cup \{s_i\} = s_j$, and $\overline{j}$ is the largest index j' such that $\max_{\hat{\succ}} \tilde{X}_{j'} \cup \{s_i\} = s_j$. Then, because of the way in which we have constructed the lottery L, $(\mu(\lambda), \tilde{\mu}(\lambda)) = (s_i, s_j)$.

Moreover, since

$$
\begin{aligned}
P(L(\lambda)) &\in P(\cup_{i',j':\underline{i} \leq i' \leq i, \underline{j} \leq j' \leq \overline{j}} \tilde{I}_{i',j'}) \\
&= \cup_{i',j':\underline{i} \leq i' \leq i, \underline{j} \leq j' \leq \overline{j}} P(\tilde{I}_{i',j'}),
\end{aligned}
$$

where $P(\tilde{I}_{i',j'}) \in [\hat{C}_{\sigma(j')}, \hat{C}_{\sigma(j'-1)}]$, it holds that under $\hat{\mu}$, student λ receives her favorite school in $\{s_i\} \cup \tilde{X}_{j'}$ for some $\underline{j} \leq j' \leq \overline{j}$, which is the school s_j. Hence $\tilde{\mu}(\lambda) = \hat{\mu}(\lambda) = s_j$.

It follows immediately that the assignment $\hat{\mu}$ is feasible, since it is equal to the feasible assignment $\tilde{\mu}$.

Finally, let us check that the assignment is stable. Suppose that $\hat{C}_j > 0$. We want to show that $\eta(\tilde{\mu}^{-1}(s_j)) = q_j$. First note that it follows from the definition of $\hat{C}_j$ that

$$
1 > \sum_{i',j':j' \leq \sigma^{-1}(j)} \gamma_{i',j'} = \sum_{i'} \rho_{i',\sigma^{-1}(j)}.
$$

Consider student preferences $\theta = (\succ, \succ)$ given by $\succ$: $[s_j, s_1, s_2, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{N+1}]$. Then $\sum_{i'} \rho_{i',\sigma^{-1}(j)}$ is the proportion of students of type θ who are assigned to school s_j in the second round, which, by assumption, is also the probability that a student with preferences θ is assigned to s_j in the second round. But since M is non-wasteful, this means that $\eta(\tilde{\mu}^{-1}(s_j)) = q_j$. It follows from constrained Pareto efficiency that the output of M is the student-optimal stable matching. □

Proof of Lemma 3. Here, we prove the prefix property. We first observe that any schools reported to be acceptable but ranked below s in the first round are inconsequential. Moreover, since M respects guarantees, weak two-round strategy-proofness implies that any schools reported to be acceptable but ranked below s in the second round are inconsequential. Hence it suffices to prove the lemma for first-round preference orderings $\succ$ and $\succ'$ for which s is the last acceptable school.

Suppose that the lemma holds for $i = i(\mathcal{C}')$. Then if $i(\mathcal{C}') = i' < i$ it holds that

$$\begin{aligned}
\rho^{\theta}\left(\left[C_j, C_i\right], \mathcal{C}'\right) &= \frac{\rho^{\theta}\left(\left[C_j, C_{i(\mathcal{C}')}\right], \mathcal{C}'\right)\left(C_{i(\mathcal{C}')} - C_j\right) - \rho^{\theta}\left(\left[C_i, C_{i(\mathcal{C}')}\right], \mathcal{C}'\right)\left(C_{i(\mathcal{C}')} - C_i\right)}{C_i - C_j} \\
&= \frac{\rho^{\theta'}\left(\left[C_j, C_{i(\mathcal{C}')}\right], \mathcal{C}'\right)\left(C_{i(\mathcal{C}')} - C_j\right) - \rho^{\theta'}\left(\left[C_i, C_{i(\mathcal{C}')}\right], \mathcal{C}'\right)\left(C_{i(\mathcal{C}')} - C_i\right)}{C_i - C_j} \\
&= \rho^{\theta'}\left(\left[C_j, C_i\right], \mathcal{C}'\right),
\end{aligned}$$

where the first and last equalities follow from Bayes' rule, and the second equality holds since the lemma holds for $i = i(\mathcal{C}')$, and the theorem follows. Hence it suffices to prove the lemma for $i = i(\mathcal{C}')$.

Let $i_1, \ldots, i_k$ be the indices of the schools in S', in increasing order. We observe that $i_k = i(S')$. Recall that $s = s_j$, where $i_k < j$.

Since we wish to prove that the lemma holds for all pairs θ, θ' satisfying the assumptions, it suffices to show that the lemma holds for a fixed preference θ when we vary only θ'. Therefore, we may, without loss of generality, fix the preferences θ to satisfy that

$$\succ: [s_{i(S')}, s_{i_1}, \ldots, s_{i_{k-1}}, s = s_j, s_{N+1}] \text{ and } \hat{\succ} : [s_{i(S')}, s_{i_1}, \cdots, s_{i_{k-1}}, s_{N+1}],$$

and all other schools are unacceptable. That is, the worst school in S' is top ranked, then all other schools in S' in order. In the first round $s = s_j$ is also acceptable, and in the second round only schools in S' are acceptable.

We remark that given the first-round ordering, the worst school in S' and the school s (namely, $s_{i(S')}$ and s_j) are the only acceptable schools to which students of type θ will be assigned in the first round. Moreover, the proportion of students with preferences θ (or θ') who can deduce that their score is in $[C_j, C_{i(S')}]$ is precisely $C_{i(S')} - C_j$, since such students are assigned in the first round to some school not in S' that they weakly prefer to s, and all such schools are between $s_{i(S'')}$ and s_j in the overdemand ordering. Similarly, the proportion of students with preferences θ (or θ') who can deduce that their lottery number is in $[C_{i(S')}, 1]$ is precisely $1 - C_{i(S')}$, since such students are assigned in the first round to a school in S'. (Note that students with preferences θ' may be able to deduce that their lottery number falls in a subinterval of the interval we have specified. However, this does not affect our statements.) To compare the proportion of students of types θ and θ' whose scores are in $[C_j, C_{i(S')}]$ and who are assigned to S' in the second round, we define a third student type θ'' as follows. Let $\theta'' = (\succ', \hat{\succ})$ be a set of preferences where the first-round preferences are the same as the first-round preferences of θ', and the second-round preferences are the same as the second-round preferences of type θ.

Let λ be a student with preferences θ, and similarly let λ' be a student with preferences θ'. We use the two-round strategy-proofness of the mechanism to show that λ has the same probability of being assigned to some school in S' in the second round as if she had reported type θ'', and similarly for λ'. Since the proportion of students of either type being assigned to a school in S' in the first round is the same and the mechanism respects guarantees, this is sufficient to prove the prefix property.

Formally, let ρ be the probability that λ is assigned to some school in S' in the second round if she reports truthfully, conditional on being able to deduce that her first-round score is in $[C_j, C_{i(S')}]$, and let ρ' be the probability that λ' is assigned to some school in S' in the second round if she reports truthfully, conditional on being able to deduce that her first-round score is in $[C_j, C_{i(S')}]$. (We note that given her first-round assignment $\mu(\rho')$, the student ρ' may actually be able to deduce more about her first-round score, and so the interim probability after knowing her assignment that ρ' is assigned to some school in S' in the second round if she reports truthfully is not necessarily ρ'.) Let ρ'' be the probability that a student with preferences θ'' and a first-round score in $[C_j, C_{i(S')}]$ chosen uniformly at random is assigned to some school in S' in the second round. It follows from the design of the first round and from anonymity that ρ is the probability that a student with preferences θ and a lottery number in $[C_j, C_{i(S')}]$ chosen uniformly at random is assigned to some school in S' in the second round, and similarly for ρ'.

Proving the lemma is equivalent to proving $\rho = \rho'$. We show that $\rho = \rho'' = \rho'$. Note that the first equality is between preferences that are identical in the second round, and the second equality is between preferences that are identical in the first round.

We first show that $\rho = \rho''$; that is, changing just the first-round preferences does not affect the probability of assignment to S'. This is almost immediate from first-order stochastic dominance of truthful reporting, since the second-round preferences under θ and θ'' are identical. (This also illustrates the power of the assumption that the second-round assignment does not depend on first-round preferences. It implies that manipulating first-round reports to obtain a more fine-grained knowledge of the lottery number does not help, since assignment probabilities are conditionally independent of the lottery number.) We present the full argument below.

Let π be the probability that a student with preferences θ who is unassigned in the first round is assigned to a school in S' in the second round. We note that since the last acceptable school under preferences θ and θ' is $s = s_j$, the set of students with preferences θ who are unassigned in the first round is equal to the set of students with preferences θ with lottery number in $[0, C_j]$, and similarly the set of students with preferences θ'' who are unassigned in the first round is equal to the set of students with preferences θ'' with lottery number in $[0, C_j]$. Hence, the fact that θ and θ'' have the same second preferences gives us that π is also the probability that a student with preferences θ'' who is unassigned in the first round is assigned to a school in S' in the second round.

The probability of being assigned in the second round to a school in S' when reporting θ is given by:

$$(1 - C_{i(S')}) + (C_{i(S')} - C_j)\rho + C_j\pi,$$

The probability of being assigned in the second round to a school in S' when reporting θ'' is given by:

$$(1 - C_{i(S')}) + (C_{i(S')} - C_j)\rho'' + C_j\pi,$$

It follows from first-order stochastic dominance of truthful reporting for types θ and θ' that $\rho = \rho''$.

We now show that $\rho' = \rho''$. This is a little more involved, but essentially relies on breaking the set of students with first-round score in $[C_j, C_{i(S')}]$ into smaller subsets, depending on their first-round assignment, and using first-order stochastic dominance of truthful reporting to show that in each subset, the probability of an arbitrary student being assigned to a school in S' in the second round is the same for students with either set of preferences θ' or θ''.

We first introduce some notation for describing the first-round preferences of θ' and θ''. Let $\{j_1 \leq \cdots \leq j_m\}$ be the indices between $i(S')$ and j corresponding to schools that a student with preferences θ' and a lottery number in $[C_j, C_{i(S')}]$ could have been assigned to in the first round. Formally, we define them to be the indices k for which $s_k \notin S'$, $i(S') < k \leq j$, $s_k \succeq' s_j$ and s_k is relevant in the first-round overdemand ordering, that is, $k' < k$ for all k' such that $s_{k'} \succ' s_k$. We observe that $j_m = j$. For $l = 1, \ldots, m$, let ρ'_l be the probability that a student with preferences θ' who was assigned to school s_{j_l} is assigned to a school in S' in the second round.

The set of students with preferences θ' assigned to school s_{j_l} in the first round is precisely the set of students with preferences θ' whose first-round lottery number is in $[C_{j_l}, C_{j_{l-1}}]$ and similarly the set of students with preferences θ'' assigned to school s_{j_l} in the first round is precisely the set of students with preferences θ'' whose first-round lottery number is in $[C_{j_l}, C_{j_{l-1}}]$. If we define $j_0 = i$, it follows that $(C_{i(S')} - C_j) = \sum_{l=1}^{m}(C_{j_{l-1}} - C_{j_l})$, and that

$$(C_{i(S')} - C_j)\rho' = \sum_{l=1}^{m}(C_{j_{l-1}} - C_{j_l})\rho'_l.$$

Let ρ''_l be the probability that a student with preferences θ'' who was assigned to school s_{j_l} is assigned to a school in S' in the second round. Then it also holds that

$$(C_{i(S')} - C_j)\rho'' = \sum_{l=1}^{m}(C_{j_{l-1}} - C_{j_l})\rho''_l.$$

We show now that $\rho''_l = \rho'_l$ for all l, which implies that $\rho' = \rho''$.

Consider a student λ_l who reported $\succ'=\succ''$ in the first round and was assigned to school s_{j_l}. Note that such a report is consistent with either reporting θ' or θ'', and since the first-round reports of these types are the same and the first-round mechanism is DA-STB there exists some set of lottery numbers L_l such that students of type θ' and θ'' are assigned to j_l in the first round if and only if their lottery lies in L_l. The probabilities that this student is assigned in the second round to a school in $\mathcal{C}'$ when reporting θ' and θ'' are given by $\rho'{}_l$ and $\rho''{}_l$ respectively. Now for any fixed lottery $L(\lambda)$, truthful reporting is a dominant strategy in the second round for types θ and θ'. It follows that $\rho'{}_l = \rho''_l$. This completes the proof of the lemma. $\square$